CHAPTER 5 Applications of the Integral

5.1 Concepts Review

1. $\int_a^b f(x)dx; -\int_a^b f(x)dx$

2. slice, approximate, integrate

3. $g(x)-f(x);\ f(x)=g(x)$

4. $\int_c^d [q(y)-p(y)]dy$

Problem Set 5.1

1. Slice vertically.
$\Delta A \approx (x^2+1)\Delta x$
$A=\int_{-1}^2 (x^2+1)dx = \left[\frac{1}{3}x^3+x\right]_{-1}^2 = 6$

2. Slice vertically.
$\Delta A \approx (x^3-x+2)\Delta x$
$A=\int_{-1}^2 (x^3-x+2)dx = \left[\frac{1}{4}x^4-\frac{1}{2}x^2+2x\right]_{-1}^2 = \frac{33}{4}$

3. Slice vertically.
$\Delta A \approx \left[(x^2+2)-(-x)\right]\Delta x = (x^2+x+2)\Delta x$
$A=\int_{-2}^2 (x^2+x+2)dx = \left[\frac{1}{3}x^3+\frac{1}{2}x^2+2x\right]_{-2}^2$
$=\left(\frac{8}{3}+2+4\right)-\left(-\frac{8}{3}+2-4\right)=\frac{40}{3}$

4. Slice vertically.
$\Delta A \approx -(x^2+2x-3)\Delta x = (-x^2-2x+3)\Delta x$
$A=\int_{-3}^1 (-x^2-2x+3)dx = \left[-\frac{1}{3}x^3-x^2+3x\right]_{-3}^1 = \frac{32}{3}$

5. To find the intersection points, solve $2-x^2=x$.
$x^2+x-2=0$
$(x+2)(x-1)=0$
$x=-2, 1$
Slice vertically.
$\Delta A \approx \left[(2-x^2)-x\right]\Delta x = (-x^2-x-2)\Delta x$
$A=\int_{-2}^1 (-x^2-x+2)dx = \left[-\frac{1}{3}x^3-\frac{1}{2}x^2+2x\right]_{-2}^1$
$=\left(-\frac{1}{3}-\frac{1}{2}+2\right)-\left(\frac{8}{3}-2-4\right)=\frac{9}{2}$

6. To find the intersection points, solve
$x+4=x^2-2$.
$x^2-x-6=0$
$(x+2)(x-3)=0$
$x=-2, 3$
Slice vertically.
$\Delta A \approx \left[(x+4)-(x^2-2)\right]\Delta x = (-x^2+x+6)\Delta x$
$A=\int_{-2}^3 (-x^2+x+6)\,dx = \left[-\frac{1}{3}x^3+\frac{1}{2}x^2+6x\right]_{-2}^3$
$\left(-9+\frac{9}{2}+18\right)-\left(\frac{8}{3}+2-12\right)=\frac{125}{6}$

7. Solve $x^3-x^2-6x=0$.
$x(x^2-x-6)=0$
$x(x+2)(x-3)=0$
$x=-2, 0, 3$
Slice vertically.
$\Delta A_1 \approx (x^3-x^2-6x)\Delta x$
$\Delta A_2 \approx -(x^3-x^2-6x)\Delta x = (-x^3+x^2+6x)\Delta x$
$A=A_1+A_2$
$=\int_{-2}^0 (x^3-x^2-6x)dx+\int_0^3 (-x^3+x^2+6x)dx$
$=\left[\frac{1}{4}x^4-\frac{1}{3}x^3-3x^2\right]_{-2}^0+\left[-\frac{1}{4}x^4+\frac{1}{3}x^3+3x^2\right]_0^3$
$=\left[0-\left(4+\frac{8}{3}-12\right)\right]+\left[-\frac{81}{4}+9+27-0\right]$
$=\frac{16}{3}+\frac{63}{4}=\frac{253}{12}$

8. To find the intersection points, solve
$-x+2=x^2$.
$x^2+x-2=0$
$(x+2)(x-1)=0$
$x=-2, 1$
Slice vertically.
$\Delta A \approx \left[(-x+2)-x^2\right]\Delta x = (-x^2-x+2)\Delta x$
$A=\int_{-2}^1 (-x^2-x+2)dx = \left[-\frac{1}{3}x^3-\frac{1}{2}x^2+2x\right]_{-2}^1$
$=\left(-\frac{1}{3}-\frac{1}{2}+2\right)-\left(\frac{8}{3}-2-4\right)=\frac{9}{2}$

9. To find the intersection points, solve

$y+1=3-y^2$.

$y^2+y-2=0$

$(y+2)(y-1)=0$

$y=-2, 1$

Slice horizontally.

$$\Delta A \approx \left[(3-y^2)-(y+1)\right]\Delta y=(-y^2-y+2)\Delta y$$

$$A=\int_{-2}^{1}(-y^2-y+2)dy=\left[-\frac{1}{3}y^3-\frac{1}{2}y^2+2y\right]_{-2}^{1}$$

$$=\left(-\frac{1}{3}-\frac{1}{2}+2\right)-\left(\frac{8}{3}-2-4\right)=\frac{9}{2}$$

10. To find the intersection point, solve $y^2=6-y$.

$y^2+y-6=0$

$(y+3)(y-2)=0$

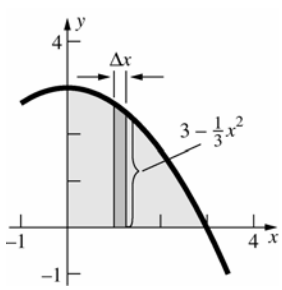
$y=-3, 2$

Slice horizontally.

$$\Delta A \approx \left[(6-y)-y^2\right]\Delta y=(-y^2-y+6)\Delta y$$

$$A=\int_{0}^{2}(-y^2-y+6)dy=\left[-\frac{1}{3}y^3-\frac{1}{2}y^2+6y\right]_{0}^{2}=\frac{22}{3}$$

11.

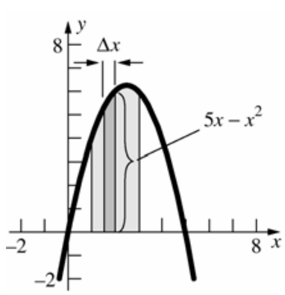


$$\Delta A \approx \left(3-\frac{1}{3}x^2\right)\Delta x$$

$$A=\int_{0}^{3}\left(3-\frac{1}{3}x^2\right)dx=\left[3x-\frac{1}{9}x^3\right]_{0}^{3}=9-3=6$$

Estimate the area to be (3)(2) = 6.

12.

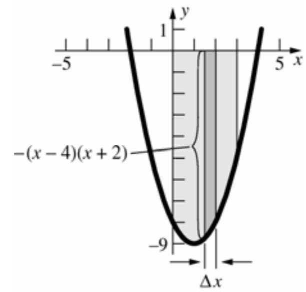


$$\Delta A \approx (5x-x^2)\Delta x$$

$$A=\int_{1}^{3}(5x-x^2)dx=\left[\frac{5}{2}x^2-\frac{1}{3}x^3\right]_{1}^{3}\approx 11.33$$

Estimate the area to be $(2)\left(5\frac{1}{2}\right)=11$.

13.



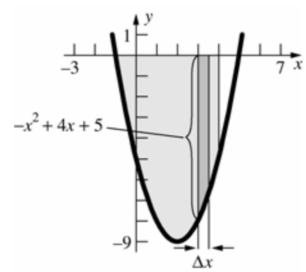
$$\Delta A \approx -(x-4)(x+2)\Delta x=(-x^2+2x+8)\Delta x$$

$$A=\int_{0}^{3}(-x^2+2x+8)dx=\left[-\frac{1}{3}x^3+x^2+8x\right]_{0}^{3}$$

$= -9 + 9 + 24 = 24$

Estimate the area to be (3)(8) = 24.

14.



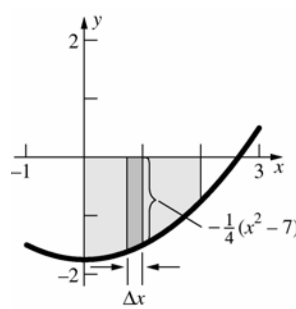
$$\Delta A \approx -(x^2-4x-5)\Delta x=(-x^2+4x+5)\Delta x$$

$$A=\int_{-1}^{4}(-x^2+4x+5)dx=\left[-\frac{1}{3}x^3+2x^2+5x\right]_{-1}^{4}$$

$$=\left(-\frac{64}{3}+32+20\right)-\left(\frac{1}{3}+2-5\right)=\frac{100}{3}\approx 33.33$$

Estimate the area to be $(5)\left(6\frac{1}{2}\right)=32\frac{1}{2}$.

15.

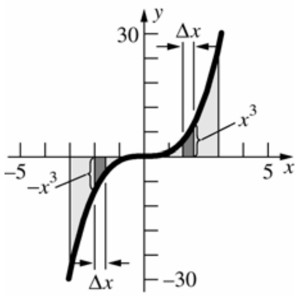


$$\Delta A \approx -\frac{1}{4}(x^2-7)\Delta x$$

$$A=\int_{0}^{2}-\frac{1}{4}(x^2-7)dx=-\frac{1}{4}\left[\frac{1}{3}x^3-7x\right]_{0}^{2}$$

$$=-\frac{1}{4}\left(\frac{8}{3}-14\right)=\frac{17}{6}\approx 2.83$$

Estimate the area to be $(2)\left(1\frac{1}{2}\right)=3$.

16. 

$\Delta A_1 \approx -x^3 \Delta x$

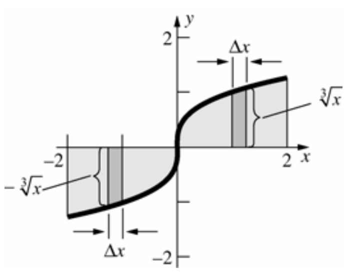
$\Delta A_2 \approx x^3 \Delta x$

$A = A_1 + A_2 = \int_{-3}^{0} -x^3 dx + \int_{0}^{3} x^3 dx$

$= \left[-\frac{1}{4}x^4\right]_{-3}^{0} + \left[\frac{1}{4}x^4\right]_{0}^{3} = \left(\frac{81}{4}\right) + \left(\frac{81}{4}\right) = \frac{81}{2}$

$= 40.5$

Estimate the area to be (3)(7) + (3)(7) = 42.

17. 

$\Delta A_1 \approx -\sqrt[3]{x}\, \Delta x$

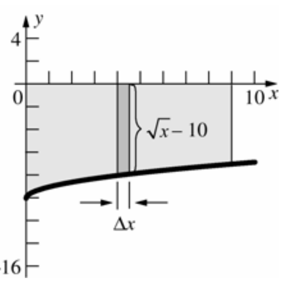
$\Delta A_2 \approx \sqrt[3]{x}\, \Delta x$

$A = A_1 + A_2 = \int_{-2}^{0} -\sqrt[3]{x}\, dx + \int_{0}^{2} \sqrt[3]{x}\, dx$

$= \left[-\frac{3}{4}x^{4/3}\right]_{-2}^{0} + \left[\frac{3}{4}x^{4/3}\right]_{0}^{2} = \left(\frac{3\sqrt[3]{2}}{2}\right) + \left(\frac{3\sqrt[3]{2}}{2}\right)$

$= 3\sqrt[3]{2} \approx 3.78$

Estimate the area to be (2)(1) + (2)(1) = 4.

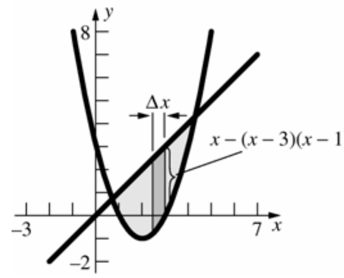
18. 

$\Delta A \approx -(\sqrt{x} - 10)\Delta x = (10 - \sqrt{x})\Delta x$

$A = \int_0^9 (10 - \sqrt{x})\, dx = \left[10x - \frac{2}{3}x^{3/2}\right]_0^9$

$= 90 - 18 = 72$

Estimate the area to be 9 · 8 =72.

19. 

$\Delta A \approx \left[x - (x-3)(x-1)\right]\Delta x$

$= \left[x - (x^2 - 4x + 3)\right]\Delta x = (-x^2 + 5x - 3)\Delta x$

To find the intersection points, solve
$x = (x - 3)(x - 1)$.

$x^2 - 5x + 3 = 0$

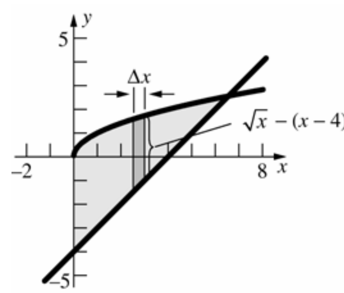
$x = \frac{5 \pm \sqrt{25 - 12}}{2}$

$x = \frac{5 \pm \sqrt{13}}{2}$

$A = \int_{\frac{5-\sqrt{13}}{2}}^{\frac{5+\sqrt{13}}{2}} (-x^2 + 5x - 3)dx$

$= \left[-\frac{1}{3}x^3 + \frac{5}{2}x^2 - 3x\right]_{\frac{5-\sqrt{13}}{2}}^{\frac{5+\sqrt{13}}{2}} = \frac{13\sqrt{13}}{6} \approx 7.81$

Estimate the area to be $\frac{1}{2}(4)(4) = 8$.

20. 

$\Delta A \approx \left[\sqrt{x} - (x-4)\right]\Delta x = \left(\sqrt{x} - x + 4\right)\Delta x$

To find the intersection point, solve
$\sqrt{x} = (x - 4)$.

$x = (x - 4)^2$

$x^2 - 9x + 16 = 0$

$x = \frac{9 \pm \sqrt{81 - 64}}{2}$

$x = \frac{9 \pm \sqrt{17}}{2}$

$\left(x = \frac{9 - \sqrt{17}}{2} \text{ is extraneous so } x = \frac{9 + \sqrt{17}}{2}.\right)$

$$A = \int_0^{\frac{9+\sqrt{17}}{2}} \left(\sqrt{x} - x + 4\right) dx$$

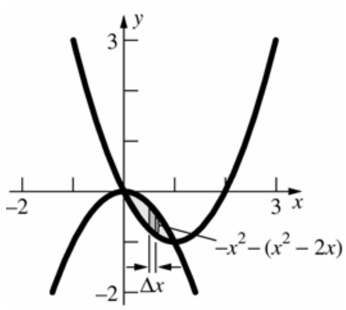
$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 4x\right]_0^{\frac{9+\sqrt{17}}{2}}$$

$$= \frac{2}{3}\left(\frac{9+\sqrt{17}}{2}\right)^{3/2} - \frac{1}{2}\left(\frac{9+\sqrt{17}}{2}\right)^2 + 4\left(\frac{9+\sqrt{17}}{2}\right)$$

$$= \frac{2}{3}\left(\frac{9+\sqrt{17}}{2}\right)^{3/2} + \frac{23}{4} - \frac{\sqrt{17}}{4} \approx 15.92$$

Estimate the area to be $\frac{1}{2}\left(5\frac{1}{2}\right)\left(5\frac{1}{2}\right) = 15\frac{1}{8}$.

21.



$$\Delta A \approx \left[-x^2 - (x^2 - 2x)\right]\Delta x = (-2x^2 + 2x)\Delta x$$

To find the intersection points, solve

$-x^2 = x^2 - 2x$.

$2x^2 - 2x = 0$

$2x(x - 1) = 0$

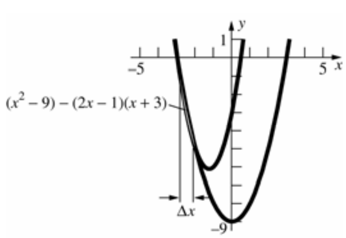
$x = 0, x = 1$

$$A = \int_0^1 (-2x^2 + 2x)dx = \left[-\frac{2}{3}x^3 + x^2\right]_0^1$$

$$= -\frac{2}{3} + 1 = \frac{1}{3} \approx 0.33$$

Estimate the area to be $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$.

22.



$$\Delta A \approx \left[(x^2 - 9) - (2x - 1)(x + 3)\right]\Delta x$$

$$= \left[(x^2 - 9) - (2x^2 + 5x - 3)\right]\Delta x$$

$$= (-x^2 - 5x - 6)\Delta x$$

To find the intersection points, solve

$(2x - 1)(x + 3) = x^2 - 9$.

$x^2 + 5x + 6 = 0$

$(x + 3)(x + 2) = 0$

$x = -3, -2$

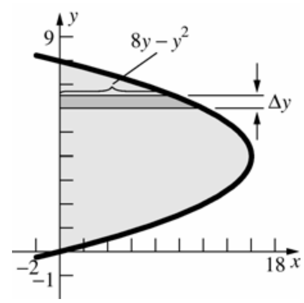
$$A = \int_{-3}^{-2} (-x^2 - 5x - 6)dx$$

$$= \left[-\frac{1}{3}x^3 - \frac{5}{2}x^2 - 6x\right]_{-3}^{-2}$$

$$= \left(\frac{8}{3} - 10 + 12\right) - \left(9 - \frac{45}{2} + 18\right) = \frac{1}{6} \approx 0.17$$

Estimate the area to be $\frac{1}{2}(1)\left(5 - 4\frac{2}{3}\right) = \frac{1}{6}$.

23.



$$\Delta A \approx (8y - y^2)\Delta y$$

To find the intersection points, solve

$8y - y^2 = 0$.

$y(8 - y) = 0$

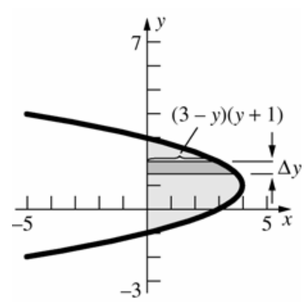
$y = 0, 8$

$$A = \int_0^8 (8y - y^2)\, dy = \left[4y^2 - \frac{1}{3}y^3\right]_0^8$$

$$= 256 - \frac{512}{3} = \frac{256}{3} \approx 85.33$$

Estimate the area to be (16)(5) = 80.

24.



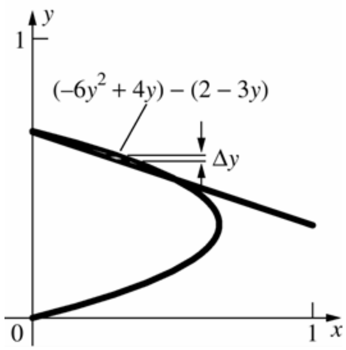
$$\Delta A \approx (3 - y)(y + 1)\Delta y = (-y^2 + 2y + 3)\Delta y$$

$$A = \int_{-1}^3 (-y^2 + 2y + 3)dy = \left[-\frac{1}{3}y^3 + y^2 + 3y\right]_{-1}^3$$

$$= (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3\right) = \frac{32}{3} \approx 10.67$$

Estimate the area to be $(4)\left(2\frac{1}{2}\right) = 10$.

25.



$$\Delta A \approx \left[(-6y^2+4y)-(2-3y)\right]\Delta y$$

$$= (-6y^2+7y-2)\Delta y$$

To find the intersection points, solve

$-6y^2+4y = 2-3y.$

$6y^2-7y+2=0$

$(2y-1)(3y-2)=0$

$y=\frac{1}{2},\frac{2}{3}$

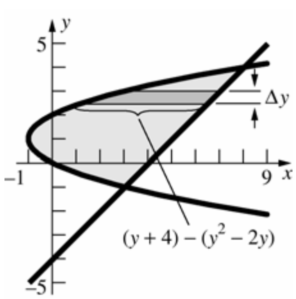
$$A=\int_{1/2}^{2/3}(-6y^2+7y-2)dy=\left[-2y^3+\frac{7}{2}y^2-2y\right]_{1/2}^{2/3}$$

$$=\left(-\frac{16}{27}+\frac{14}{9}-\frac{4}{3}\right)-\left(-\frac{1}{4}+\frac{7}{8}-1\right)=\frac{1}{216}\approx 0.0046$$

Estimate the area to be

$\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{5}\right)-\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)=\frac{1}{120}$.

26.



$$\Delta A \approx \left[(y+4)-(y^2-2y)\right]\Delta y=(-y^2+3y+4)\Delta y$$

To find the intersection points, solve

$y^2-2y=y+4$.

$y^2-3y-4=0$

$(y+1)(y-4)=0$

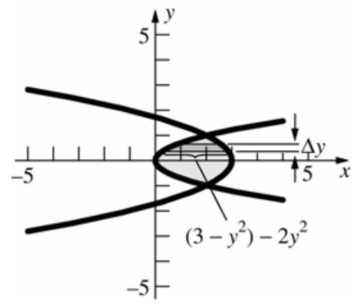
$y=-1, 4$

$$A=\int_{-1}^{4}(-y^2+3y+4)dy=\left[-\frac{1}{3}y^3+\frac{3}{2}y^2+4y\right]_{-1}^{4}$$

$$=\left(-\frac{64}{3}+24+16\right)-\left(\frac{1}{3}+\frac{3}{2}-4\right)=\frac{125}{6}\approx 20.83$$

Estimate the area to be (7)(3) = 21.

27.



$$\Delta A \approx \left[(3-y^2)-2y^2\right]\Delta y=(-3y^2+3)\Delta y$$

To find the intersection points, solve

$2y^2=3-y^2$.

$3y^2-3=0$

$3(y+1)(y-1)=0$

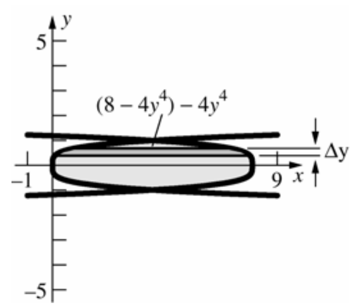
$y=-1, 1$

$$A=\int_{-1}^{1}(-3y^2+3)dy=\left[-y^3+3y\right]_{-1}^{1}$$

$= (-1+3)-(1-3)=4$

Estimate the value to be (2)(2) = 4.

28.



$$\Delta A \approx \left[(8-4y^4)-(4y^4)\right]\Delta y=(8-8y^4)\Delta y$$

To find the intersection points, solve

$4y^4=8-4y^4$.

$8y^4=8$

$y^4=1$

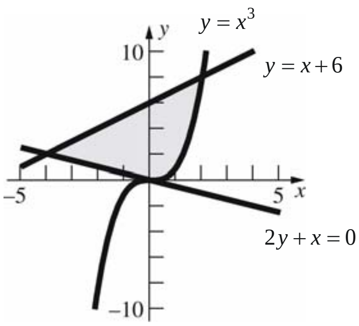
$y=\pm 1$

$$A=\int_{-1}^{1}(8-8y^4)dy=\left[8y-\frac{8}{5}y^5\right]_{-1}^{1}$$

$$=\left(8-\frac{8}{5}\right)-\left(-8+\frac{8}{5}\right)=\frac{64}{5}=12.8$$

Estimate the area to be $(8)\left(1\frac{1}{2}\right)=12$.

29.



Let R_1 be the region bounded by $2y + x = 0$, $y = x + 6$, and $x = 0$.

$$A(R_1) = \int_{-4}^{0} \left[(x+6) - \left(-\frac{1}{2}x \right) \right] dx$$

$$= \int_{-4}^{0} \left(\frac{3}{2}x + 6 \right) dx$$

Let R_2 be the region bounded by $y = x + 6$, $y = x^3$, and $x = 0$.

$$A(R_2) = \int_0^2 \left[(x+6) - x^3 \right] dx = \int_0^2 (-x^3 + x + 6)dx$$

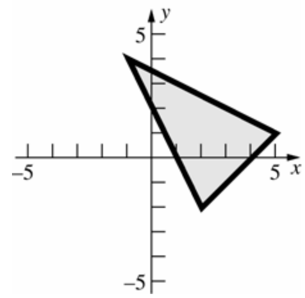
$$A(R) = A(R_1) + A(R_2)$$

$$= \int_{-4}^{0} \left(\frac{3}{2}x + 6 \right) dx + \int_0^2 (-x^3 + x + 6)dx$$

$$= \left[\frac{3}{4}x^2 + 6x \right]_{-4}^{0} + \left[-\frac{1}{4}x^4 + \frac{1}{2}x^2 + 6x \right]_0^2$$

$= 12 + 10 = 22$

30.



An equation of the line through (–1, 4) and (5, 1) is $y = -\frac{1}{2}x + \frac{7}{2}$. An equation of the line through (–1, 4) and (2, –2) is $y = –2x + 2$. An equation of the line through (2, –2) and (5, 1) is $y = x – 4$. Two integrals must be used. The left-hand part of the triangle has area

$$\int_{-1}^{2} \left[-\frac{1}{2}x + \frac{7}{2} - (-2x + 2) \right] dx = \int_{-1}^{2} \left(\frac{3}{2}x + \frac{3}{2} \right) dx .$$

The right-hand part of the triangle has area

$$\int_2^5 \left[-\frac{1}{2}x + \frac{7}{2} - (x - 4) \right] dx = \int_2^5 \left(-\frac{3}{2}x + \frac{15}{2} \right) dx .$$

The triangle has area

$$\int_{-1}^{2} \left(\frac{3}{2}x + \frac{3}{2} \right) dx + \int_2^5 \left(-\frac{3}{2}x + \frac{15}{2} \right) dx$$

$$= \left[\frac{3}{4}x^2 + \frac{3}{2}x \right]_{-1}^{2} + \left[-\frac{3}{4}x^2 + \frac{15}{2}x \right]_2^5$$

$$= \frac{27}{4} + \frac{27}{4} = \frac{27}{2} = 13.5$$

31. $\int_{-1}^{9} (3t^2 - 24t + 36)dt = \left[t^3 - 12t^2 + 36t \right]_{-1}^{9}$ = (729 – 972 + 324) – (–1 – 12 – 36) = 130

The displacement is 130 ft. Solve $3t^2 - 24t + 36 = 0$.

$3(t – 2)(t – 6) = 0$

$t = 2, 6$

$$|V(t)| = \begin{cases} 3t^2 - 24t + 36 & t \le 2, t \ge 6 \\ -3t^2 + 24t - 36 & 2 < t < 6 \end{cases}$$

$$\int_{-1}^{9} \left| 3t^2 - 24t + 36 \right| dt = \int_{-1}^{2} (3t^2 - 24t + 36)\, dt + \int_2^6 (-3t^2 + 24t - 36)\, dt + \int_6^9 (3t^2 - 24t + 36)\, dt$$

$$= \left[t^3 - 12t^2 + 36t \right]_{-1}^{2} + \left[-t^3 + 12t^2 - 36t \right]_2^6 + \left[t^3 - 12t^2 + 36t \right]_6^9 = 81 + 32 + 81 = 194$$

The total distance traveled is 194 feet.

32. $\int_0^{3\pi/2} \left(\frac{1}{2} + \sin 2t \right) dt = \left[\frac{1}{2}t - \frac{1}{2}\cos 2t \right]_0^{3\pi/2} = \left(\frac{3\pi}{4} + \frac{1}{2} \right) - \left(0 - \frac{1}{2} \right) = \frac{3\pi}{4} + 1$

The displacement is $\frac{3\pi}{4} + 1 \approx 3.36$ feet . Solve $\frac{1}{2} + \sin 2t = 0$ for $0 \le t \le \frac{3\pi}{2}$.

$$\sin 2t = -\frac{1}{2} \Rightarrow 2t = \frac{7\pi}{6}, \frac{11\pi}{6} \Rightarrow t = \frac{7\pi}{12}, \frac{11\pi}{12}$$

$$\left|\frac{1}{2}+\sin 2t\right| = \begin{cases} \frac{1}{2}+\sin 2t & 0 \le t \le \frac{7\pi}{12}, \frac{11\pi}{12} \le t \le \frac{3\pi}{2} \\ -\frac{1}{2}-\sin 2t & \frac{7\pi}{12} < t < \frac{11\pi}{12} \end{cases}$$

$$\int_0^{3\pi/2}\left|\frac{1}{2}+\sin 2t\right|dt = \int_0^{7\pi/12}\left(\frac{1}{2}+\sin 2t\right)dt + \int_{7\pi/12}^{11\pi/12}\left(-\frac{1}{2}-\sin 2t\right)dt + \int_{11\pi/12}^{3\pi/2}\left(\frac{1}{2}+\sin 2t\right)dt$$

$$= \left[\frac{1}{2}t-\frac{1}{2}\cos 2t\right]_0^{7\pi/12} + \left[-\frac{1}{2}t+\frac{1}{2}\cos 2t\right]_{7\pi/12}^{11\pi/12} + \left[\frac{1}{2}t-\frac{1}{2}\cos 2t\right]_{11\pi/12}^{3\pi/2}$$

$$= \left(\frac{7\pi}{24}+\frac{\sqrt{3}}{4}+\frac{1}{2}\right)+\left(-\frac{\pi}{6}+\frac{\sqrt{3}}{2}\right)+\left(\frac{7\pi}{24}+\frac{\sqrt{3}}{4}+\frac{1}{2}\right) = \frac{5\pi}{12}+\sqrt{3}+1$$

The total distance traveled is $\frac{5\pi}{12}+\sqrt{3}+1 \approx 4.04$ feet.

33. $s(t) = \int v(t)dt = \int (2t-4)dt = t^2 - 4t + C$

Since $s(0) = 0$, $C = 0$ and $s(t) = t^2 - 4t$. $s = 12$ when $t = 6$, so it takes the object 6 seconds to get $s = 12$.

$$|2t-4| = \begin{cases} 4-2t & 0 \le t < 2 \\ 2t-4 & 2 \le t \end{cases}$$

$\int_0^2 |2t-4|\,dt = \left[-t^2+4t\right]_0^2 = 4,$ so the object travels a distance of 4 cm in the first two seconds.

$$\int_2^x |2t-4|\,dt = \left[t^2-4t\right]_2^x = x^2-4x+4$$

$x^2-4x+4 = 8$ when $x = 2+2\sqrt{2}$, so the object takes $2+2\sqrt{2} \approx 4.83$ seconds to travel a total distance of 12 centimeters.

34. a. $A = \int_1^6 x^{-2}dx = \left[-\frac{1}{x}\right]_1^6 = -\frac{1}{6}+1 = \frac{5}{6}$

b. Find c so that $\int_1^c x^{-2}dx = \frac{5}{12}$.

$$\int_1^c x^{-2}dx = \left[-\frac{1}{x}\right]_1^c = 1-\frac{1}{c}$$

$$1-\frac{1}{c} = \frac{5}{12}, c = \frac{12}{7}$$

The line $x = \frac{12}{7}$ bisects the area.

c. Slicing the region horizontally, the area is $\int_{1/36}^1 \frac{1}{\sqrt{y}}dy + \left(\frac{1}{36}\right)(5)$. Since $\frac{5}{36} < \frac{5}{12}$ the line that bisects the area is between $y = \frac{1}{36}$ and $y = 1$, so we find d such that

$$\int_d^1 \frac{1}{\sqrt{y}}dy = \frac{5}{12}; \int_d^1 \frac{1}{\sqrt{y}}dy = \left[2\sqrt{y}\right]_d^1$$

$$= 2-2\sqrt{d}\ ;\ 2-2\sqrt{d} = \frac{5}{12};$$

$$d = \frac{361}{576} \approx 0.627.$$

The line $y = 0.627$ approximately bisects the area.

35. Equation of line through (–2, 4) and (3, 9): $y = x + 6$
Equation of line through (2, 4) and (–3, 9): $y = -x + 6$

$$A(A) = \int_{-3}^0 [9-(-x+6)]dx + \int_0^3 [9-(x+6)]dx$$

$$= \int_{-3}^0 (3+x)dx + \int_0^3 (3-x)dx$$

$$= \left[3x+\frac{1}{2}x^2\right]_{-3}^0 + \left[3x-\frac{1}{2}x^2\right]_0^3 = \frac{9}{2}+\frac{9}{2} = 9$$

$$A(B) = \int_{-3}^{-2}[(-x+6)-x^2]dx + \int_{-2}^0 [(-x+6)-(x+6)]dx$$

$$= \int_{-3}^{-2}(-x^2-x+6)dx + \int_{-2}^0 (-2x)dx$$

$$= \left[-\frac{1}{3}x^3-\frac{1}{2}x^2+6x\right]_{-3}^{-2} + \left[-x^2\right]_{-2}^0 = \frac{37}{6}$$

$$A(C) = A(B) = \frac{37}{6} \text{ (by symmetry)}$$

$$A(D) = \int_{-2}^{0} [(x+6) - x^2]dx + \int_{0}^{2} [(-x+6) - x^2]dx$$

$$= \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x\right]_{-2}^{0} + \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x\right]_{0}^{2}$$

$$= \frac{44}{3}$$

$A(A) + A(B) + A(C) + A(D) = 36$

$$A(A+B+C+D) = \int_{-3}^{3} (9 - x^2)dx = \left[9x - \frac{1}{3}x^3\right]_{-3}^{3}$$

$= 36$

36. Let $f(x)$ be the width of region 1 at every x.

$\Delta A_1 \approx f(x)\Delta x$, so $A_1 = \int_a^b f(x)dx$.

Let $g(x)$ be the width of region 2 at every x.

$\Delta A_2 \approx g(x)\Delta x$, so $A_2 = \int_a^b g(x)dx$.

Since $f(x) = g(x)$ at every x in $[a, b]$,

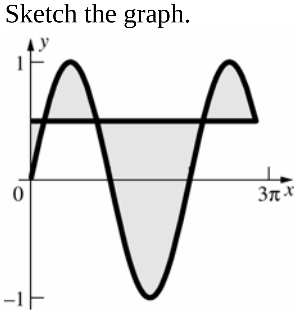
$A_1 = \int_a^b f(x)dx = \int_a^b g(x)dx = A_2$.

37. The height of the triangular region is given by for $0 \le x \le 1$. We need only show that the height of the second region is the same in order to apply Cavalieri''s Principle. The height of the second region is

$$h_2 = (x^2 - 2x + 1) - (x^2 - 3x + 1)$$
$$= x^2 - 2x + 1 - x^2 + 3x - 1$$
$$= x \text{ for } 0 \le x \le 1.$$

Since $h_1 = h_2$ over the same closed interval, we can conclude that their areas are equal.

38. Sketch the graph.



Solve $\sin x = \frac{1}{2}$ for $0 \le x \le \frac{17\pi}{6}$.

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

The area of the trapped region is

$$\int_0^{\pi/6} \left(\frac{1}{2} - \sin x\right)dx + \int_{\pi/6}^{5\pi/6} \left(\sin x - \frac{1}{2}\right)dx$$

$$+ \int_{5\pi/6}^{13\pi/6} \left(\frac{1}{2} - \sin x\right)dx + \int_{13\pi/6}^{17\pi/6} \left(\sin x - \frac{1}{2}\right)dx$$

$$= \left[\frac{1}{2}x + \cos x\right]_0^{\pi/6} + \left[-\cos x - \frac{1}{2}x\right]_{\pi/6}^{5\pi/6}$$

$$+ \left[\frac{1}{2}x + \cos x\right]_{5\pi/6}^{13\pi/6} + \left[-\cos x - \frac{1}{2}x\right]_{13\pi/6}^{17\pi/6}$$

$$= \left(\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1\right) + \left(\sqrt{3} - \frac{\pi}{3}\right) + \left(\sqrt{3} + \frac{2\pi}{3}\right)$$

$$+ \left(\sqrt{3} - \frac{\pi}{3}\right) = \frac{\pi}{12} + \frac{7\sqrt{3}}{2} - 1 \approx 5.32$$

5.2 Concepts Review

1. $\pi r^2 h$

2. $\pi(R^2 - r^2)h$

3. $\pi x^4 \Delta x$

4. $\pi[(x^2 + 2)^2 - 4]\Delta x$

Problem Set 5.2

1. Slice vertically.

$$\Delta V \approx \pi(x^2 + 1)^2 \Delta x = \pi(x^4 + 2x^2 + 1)\Delta x$$

$$V = \pi\int_0^2 (x^4 + 2x^2 + 1)dx$$

$$= \pi\left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right]_0^2 = \pi\left(\frac{32}{5} + \frac{16}{3} + 2\right) = \frac{206\pi}{15}$$

≈ 43.14

2. Slice vertically.

$$\Delta V \approx \pi(-x^2 + 4x)^2 \Delta x = \pi(x^4 - 8x^3 + 16x^2)\Delta x$$

$$V = \pi\int_0^3 (x^4 - 8x^3 + 16x^2)dx$$

$$= \pi\left[\frac{1}{5}x^5 - 2x^4 + \frac{16}{3}x^3\right]_0^3$$

$$= \pi\left(\frac{243}{5} - 162 + 144\right)$$

$$= \frac{153\pi}{5} \approx 96.13$$

3. a. Slice vertically.

$$\Delta V \approx \pi(4-x^2)^2\Delta x \;= \pi(16-8x^2+x^4)\Delta x$$

$$V = \pi\int_0^2 (16-8x^2+x^4)dx$$

$$= \frac{256\pi}{15} \approx 53.62$$

b. Slice horizontally.

$$x = \sqrt{4-y}$$

Note that when $x = 0$, $y = 4$.

$$\Delta V \approx \pi\left(\sqrt{4-y}\right)^2 \Delta y = \pi(4-y)\Delta y$$

$$V = \pi\int_0^4 (4-y)dy = \pi\left[4y-\frac{1}{2}y^2\right]_0^4$$

$$= \pi(16-8) = 8\pi \approx 25.13$$

4. a. Slice vertically.

$$\Delta V \approx \pi(4-2x)^2\Delta x$$

$$0 \le x \le 2$$

$$V = \pi\int_0^2 (4-2x)^2 dx = \pi\left[-\frac{1}{6}(4-2x)^3\right]_0^2$$

$$= \frac{32\pi}{3} \approx 33.51$$

b. Slice vertically.

$$x = 2-\frac{y}{2}$$

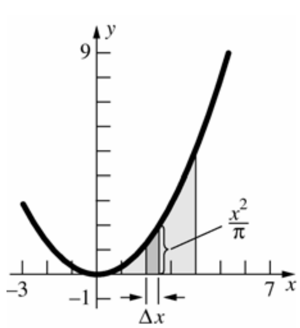
$$\Delta V \approx \pi\left(2-\frac{y}{2}\right)^2 \Delta y$$

$$0 \le y \le 4$$

$$V = \pi\int_0^4 \left(2-\frac{y}{2}\right)^2 dy = \pi\left[-\frac{2}{3}\left(2-\frac{y}{2}\right)^3\right]_0^4$$

$$= \frac{16\pi}{3} \approx 16.76$$

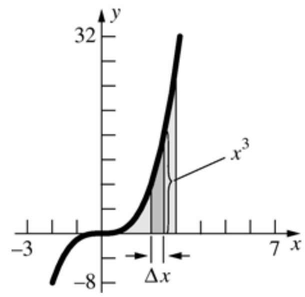
5.



$$\Delta V \approx \pi\left(\frac{x^2}{\pi}\right)^2 \Delta x = \frac{x^4}{\pi}\Delta x$$

$$V = \int_0^4 \frac{x^4}{\pi}dx = \frac{1}{\pi}\left[\frac{1}{5}x^5\right]_0^4 = \frac{1024}{5\pi} \approx 65.19$$

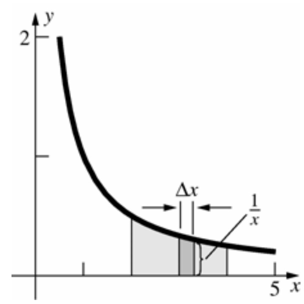
6.



$$\Delta V \approx \pi(x^3)^2\Delta x = \pi x^6\Delta x$$

$$V = \pi\int_0^3 x^6 dx = \pi\left[\frac{1}{7}x^7\right]_0^3 = \frac{2187\pi}{7} \approx 981.52$$

7.

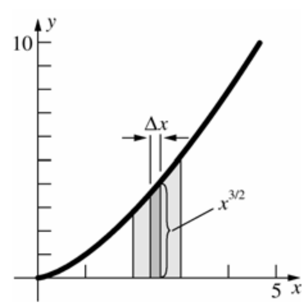


$$\Delta V \approx \pi\left(\frac{1}{x}\right)^2 \Delta x = \pi\left(\frac{1}{x^2}\right)\Delta x$$

$$V = \pi\int_2^4 \frac{1}{x^2}dx = \pi\left[-\frac{1}{x}\right]_2^4 = \pi\left(-\frac{1}{4}+\frac{1}{2}\right) = \frac{\pi}{4}$$

$$\approx 0.79$$

8.

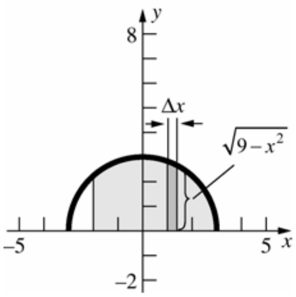


$$\Delta V \approx \pi(x^{3/2})^2\Delta x = \pi x^3\Delta x$$

$$V = \pi\int_2^3 x^3 dx = \pi\left[\frac{1}{4}x^4\right]_2^3 = \pi\left(\frac{81}{4}-\frac{16}{4}\right)$$

$$= \frac{65\pi}{4} \approx 51.05$$

9.

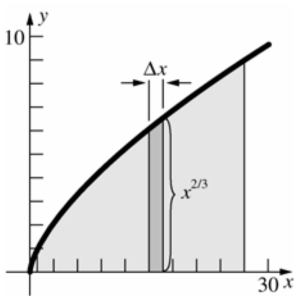


$$\Delta V \approx \pi\left(\sqrt{9-x^2}\right)^2 \Delta x = \pi(9-x^2)\Delta x$$

$$V = \pi\int_{-2}^{3} (9-x^2)dx = \pi\left[9x - \frac{1}{3}x^3\right]_{-2}^{3}$$

$$= \pi\left[(27-9) - \left(-18 + \frac{8}{3}\right)\right] = \frac{100\pi}{3} \approx 104.72$$

10.

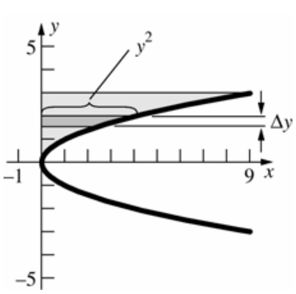


$$\Delta V \approx \pi(x^{2/3})^2 \Delta x = \pi x^{4/3}\Delta x$$

$$V = \pi\int_{1}^{27} x^{4/3}dx = \pi\left[\frac{3}{7}x^{7/3}\right]_{1}^{27} = \pi\left(\frac{6561}{7} - \frac{3}{7}\right)$$

$$= \frac{6558\pi}{7} \approx 2943.22$$

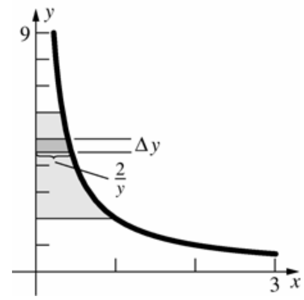
11.



$$\Delta V \approx \pi(y^2)^2 \Delta y = \pi y^4 \Delta y$$

$$V = \pi\int_{0}^{3} y^4 dy = \pi\left[\frac{1}{5}y^5\right]_{0}^{3} = \frac{243\pi}{5} \approx 152.68$$

12.

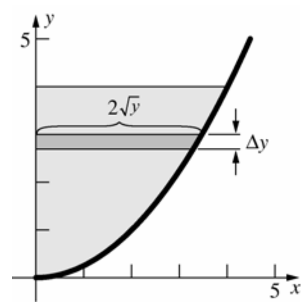


$$\Delta V \approx \pi\left(\frac{2}{y}\right)^2 \Delta y = 4\pi\left(\frac{1}{y^2}\right)\Delta y$$

$$V = 4\pi\int_{2}^{6} \frac{1}{y^2}dy = 4\pi\left[-\frac{1}{y}\right]_{2}^{6} = 4\pi\left(-\frac{1}{6} + \frac{1}{2}\right)$$

$$= \frac{4\pi}{3} \approx 4.19$$

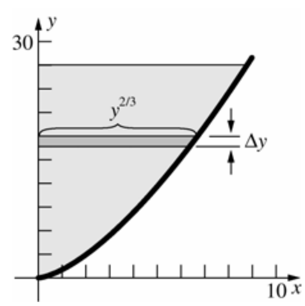
13.



$$\Delta V \approx \pi\left(2\sqrt{y}\right)^2 \Delta y = 4\pi y \Delta y$$

$$V = 4\pi\int_{0}^{4} y\,dy = 4\pi\left[\frac{1}{2}y^2\right]_{0}^{4} = 32\pi \approx 100.53$$

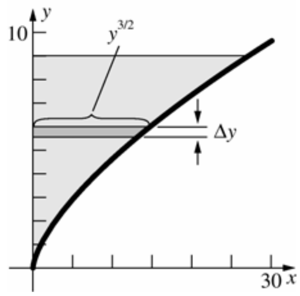
14.



$$\Delta V \approx \pi(y^{2/3})^2 \Delta y = \pi y^{4/3}\Delta y$$

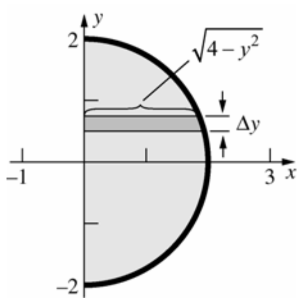
$$V = \pi\int_{0}^{27} y^{4/3}dy = \pi\left[\frac{3}{7}y^{7/3}\right]_{0}^{27} = \frac{6561\pi}{7}$$

$$\approx 2944.57$$

15. 

$$\Delta V \approx \pi(y^{3/2})^2 \Delta y = \pi y^3 \Delta y$$

$$V = \pi\int_0^9 y^3 dy = \pi\left[\frac{1}{4}y^4\right]_0^9 = \frac{6561\pi}{4} \approx 5153.00$$

16. 

$$\Delta V \approx \pi\left(\sqrt{4-y^2}\right)^2 \Delta y = \pi(4-y^2)\Delta y$$

$$V = \pi\int_{-2}^2 (4-y^2)dy = \pi\left[4y - \frac{1}{3}y^3\right]_{-2}^2$$

$$= \pi\left[\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)\right] = \frac{32\pi}{3} \approx 33.51$$

17. The equation of the upper half of the ellipse is

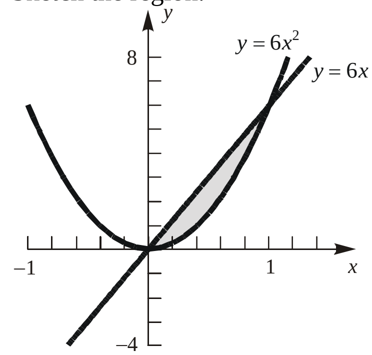
$y = b\sqrt{1-\frac{x^2}{a^2}}$ or $y = \frac{b}{a}\sqrt{a^2-x^2}$.

$$V = \pi\int_{-a}^a \frac{b^2}{a^2}(a^2-x^2)dx$$

$$= \frac{b^2\pi}{a^2}\left[a^2x - \frac{x^3}{3}\right]_{-a}^a$$

$$= \frac{b^2\pi}{a^2}\left[\left(a^3-\frac{a^3}{3}\right)-\left(-a^3+\frac{a^3}{3}\right)\right] = \frac{4}{3}ab^2\pi$$

18. Sketch the region.



To find the intersection points, solve $6x = 6x^2$.

$6(x^2 - x) = 0$

$6x(x-1) = 0$

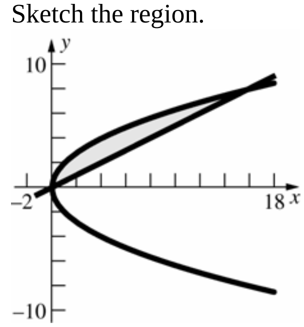
$x = 0, 1$

$$\Delta V \approx \pi\left[(6x)^2 - (6x^2)^2\right]\Delta x = 36\pi(x^2 - x^4)\Delta x$$

$$V = 36\pi\int_0^1 (x^2 - x^4)dx = 36\pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1$$

$$= 36\pi\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{24\pi}{5} \approx 15.08$$

19. Sketch the region.



To find the intersection points, solve $\frac{x}{2} = 2\sqrt{x}$.

$\frac{x^2}{4} = 4x$

$x^2 - 16x = 0$

$x(x-16) = 0$

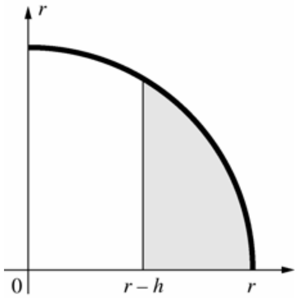
$x = 0, 16$

$$\Delta V \approx \pi\left[\left(2\sqrt{x}\right)^2 - \left(\frac{x}{2}\right)^2\right]\Delta x = \pi\left(4x - \frac{x^2}{4}\right)\Delta x$$

$$V = \pi\int_0^{16}\left(4x - \frac{x^2}{4}\right)dx = \pi\left[2x^2 - \frac{x^3}{12}\right]_0^{16}$$

$$= \pi\left(512 - \frac{1024}{3}\right) = \frac{512\pi}{3} \approx 536.17$$

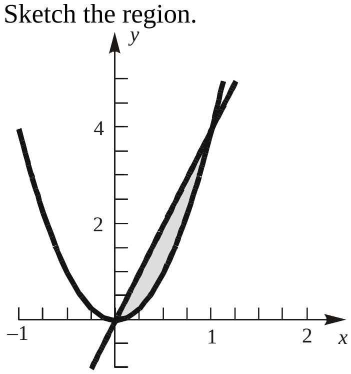
20. Sketch the region.



$$V=\pi\int_{r-h}^{r}(r^2-x^2)dx=\pi\left[r^2x-\frac{1}{3}x^3\right]_{r-h}^{r}$$

$$=\frac{1}{3}\pi h^2(3r-h)$$

21. Sketch the region.



To find the intersection points, solve $\dfrac{y}{4}=\dfrac{\sqrt{y}}{2}$.

$$\frac{y^2}{16}=\frac{y}{4}$$

$$y^2-4y=0$$

$y(y-4)=0$

$y=0, 4$

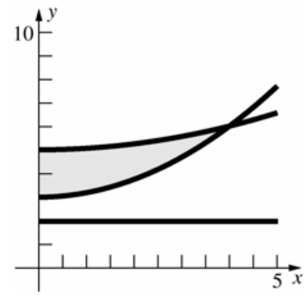
$$\Delta V\approx\pi\left[\left(\frac{\sqrt{y}}{2}\right)^2-\left(\frac{y}{4}\right)^2\right]\Delta y=\pi\left(\frac{y}{4}-\frac{y^2}{16}\right)\Delta y$$

$$V=\pi\int_0^4\left(\frac{y}{4}-\frac{y^2}{16}\right)dy=\pi\left[\frac{y^2}{8}-\frac{y^3}{48}\right]_0^4$$

$$=\frac{2\pi}{3}\approx 2.0944$$

22. $y=\dfrac{3}{16}x^2+3,\ y=\dfrac{1}{16}x^2+5$

Sketch the region.



To find the intersection point, solve

$$\frac{3}{16}x^2+3=\frac{1}{16}x^2+5.$$

$$\frac{1}{8}x^2-2=0$$

$$x^2-16=0$$

$(x+4)(x-4)=0$

$x=-4, 4$

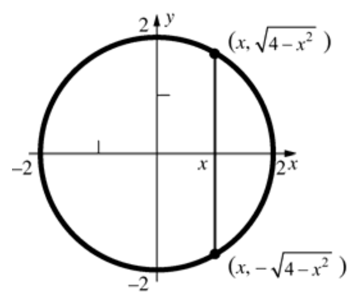
$$V=\pi\int_0^4\left[\left(\frac{1}{16}x^2+5-2\right)^2-\left(\frac{3}{16}x^2+3-2\right)^2\right]dx$$

$$=\pi\int_0^4\left[\left(\frac{1}{256}x^4-\frac{3}{8}x^2+9\right)-\left(\frac{9}{256}x^4-\frac{3}{8}x^2+1\right)\right]dx$$

$$=\pi\int_0^4\left(8-\frac{1}{32}x^4\right)dx=\pi\left[8x-\frac{1}{160}x^5\right]_0^4$$

$$=\pi\left(32-\frac{32}{5}\right)=\frac{128\pi}{5}\approx 80.42$$

23.



The square at x has sides of length $2\sqrt{4-x^2}$, as shown.

$$V=\int_{-2}^{2}\left(2\sqrt{4-x^2}\right)^2dx=\int_{-2}^{2}4(4-x^2)dx$$

$$=4\left[4x-\frac{x^3}{3}\right]_{-2}^{2}=4\left[\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)\right]=\frac{128}{3}$$

≈ 42.67

24. The area of each cross section perpendicular to the x-axis is $\frac{1}{2}(4)\left(2\sqrt{4-x^2}\right) = 4\sqrt{4-x^2}$.

The area of a semicircle with radius 2 is $\int_{-2}^{2}\sqrt{4-x^2}\,dx = 2\pi$.

$V = \int_{-2}^{2} 4\sqrt{4-x^2}\,dx = 4(2\pi) = 8\pi \approx 25.13$

25. The square at x has sides of length $\sqrt{\cos x}$.

$V = \int_{-\pi/2}^{\pi/2} \cos x dx = [\sin x]_{-\pi/2}^{\pi/2} = 2$

26. The area of each cross section perpendicular to the x-axis is $[(1-x^2)-(1-x^4)]^2 = x^8 - 2x^6 + x^4$.

$V = \int_{-1}^{1}(x^8 - 2x^6 + x^4)dx$

$= \left[\frac{1}{9}x^9 - \frac{2}{7}x^7 + \frac{1}{5}x^5\right]_{-1}^{1} = \frac{16}{315} \approx 0.051$

27. The square at x has sides of length $\sqrt{1-x^2}$.

$V = \int_0^1 (1-x^2)dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} \approx 0.67$

28. From Problem 27 we see that horizontal cross sections of one octant of the common region are squares. The length of a side at height y is $\sqrt{r^2-y^2}$ where r is the common radius of the cylinders. The volume of the "+" can be found by adding the volumes of each cylinder and subtracting off the volume of the common region (which is counted twice). The volume of one octant of the common region is

$$\int_0^r (r^2-y^2)dy = r^2y - \frac{1}{3}y^2 \Big|_0^r$$
$$= r^3 - \frac{1}{3}r^3 = \frac{2}{3}r^3$$

Thus, the volume of the "+" is

$V =$ vol. of cylinders - vol. of common region

$=2(\pi \mathrm{r}^2 l) - 8\left(\frac{2}{3}r^3\right)$

$= 2\pi(2^2)(12) - 8\left(\frac{2}{3}(2)^3\right) = 96\pi - \frac{128}{3}$

$\approx 258.93 \text{ in}^2$

29. Using the result from Problem 28, the volume of one octant of the common region in the "+" is

$$\int_0^r (r^2-y^2)dy = r^2y - \frac{1}{3}y^2 \Big|_0^r$$
$$= r^3 - \frac{1}{3}r^3 = \frac{2}{3}r^3$$

Thus, the volume inside the "+" for two cylinders of radius r and length L is

$V =$ vol. of cylinders - vol. of common region

$= 2(\pi r^2 L) - 8\left(\frac{2}{3}r^3\right)$

$= 2\pi r^2 L - \frac{16}{3}r^3$

30. From Problem 28, the volume of one octant of the common region is $\frac{2}{3}r^3$. We can find the volume of the "T" similarly. Since the "T" has one-half the common region of the "+" in Problem 28, the volume of the "T" is given by

$V =$ vol. of cylinders - vol. of common region

$=(\pi r^2)(L_1 + L_2) - 4\left(\frac{2}{3}r^3\right)$

With $r = 2, L_1 = 12,$ and $L_2 = 8$ (inches), the volume of the "T" is

$V =$ vol. of cylinders - vol. of common region

$=(\pi \mathrm{r}^2)(L_1 + L_2) - 4\left(\frac{2}{3}r^3\right)$

$= (\pi 2^2)(12+8) - 4\left(\frac{2}{3}2^3\right)$

$= 80\pi - \frac{64}{3} \text{ in}^3$

$\approx 229.99 \text{ in}^3$

31. From Problem 30, the general form for the volume of a "T" formed by two cylinders with the same radius is

$V =$ vol. of cylinders - vol. of common region

$=(\pi \mathrm{r}^2)(L_1 + L_2) - 4\left(\frac{2}{3}r^3\right)$

$= \pi r^2(L_1 + L_2) - \frac{8}{3}r^3$

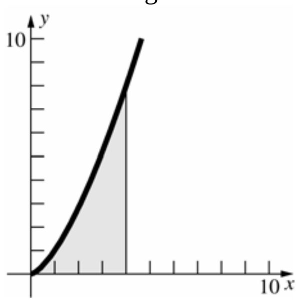
32. The area of each cross section perpendicular to the x-axis is $\frac{1}{2}\pi\left[\frac{1}{2}\left(\sqrt{x} - x^2\right)\right]^2$

$= \frac{\pi}{8}(x^4 - 2x^{5/2} + x).$

$V = \frac{\pi}{8}\int_0^1 (x^4 - 2x^{5/2} + x)dx$

$= \frac{\pi}{8}\left[\frac{1}{5}x^5 - \frac{4}{7}x^{7/2} + \frac{1}{2}x^2\right]_0^1 = \frac{9\pi}{560} \approx 0.050$

33. Sketch the region.



a. Revolving about the line $x = 4$, the radius of the disk at y is $4-\sqrt[3]{y^2} = 4-y^{2/3}$.

$V = \pi\int_0^8 (4-y^{2/3})^2\, dy$

$= \pi\int_0^8 (16-8y^{2/3}+y^{4/3})dy$

$= \pi\left[16y - \frac{24}{5}y^{5/3} + \frac{3}{7}y^{7/3}\right]_0^8$

$= \pi\left(128 - \frac{768}{5} + \frac{384}{7}\right)$

$= \frac{1024\pi}{35} \approx 91.91$

b. Revolving about the line $y = 8$, the inner radius of the disk at x is $8-\sqrt{x^3} = 8-x^{3/2}$.

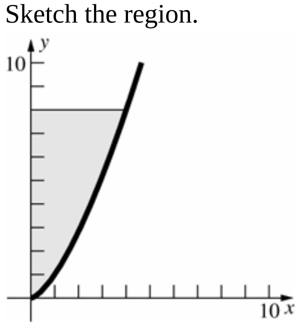
$V = \pi\int_0^4 \left[8^2 - (8-x^{3/2})^2\right]dx$

$= \pi\int_0^4 (16x^{3/2} - x^3)dx$

$= \pi\left[\frac{32}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^4 = \pi\left(\frac{1024}{5} - 64\right)$

$= \frac{704\pi}{5} \approx 442.34$

34. Sketch the region.



a. Revolving about the line $x = 4$, the inner radius of the disk at y is $4-\sqrt[3]{y^2} = 4-y^{2/3}$.

$V = \pi\int_0^8 \left[4^2 - \left(4-y^{2/3}\right)^2\right]dy$

$= \pi\int_0^8 (8y^{2/3} - y^{4/3})dy$

$= \pi\left[\frac{24}{5}y^{5/3} - \frac{3}{7}y^{7/3}\right]_0^8$

$= \pi\left(\frac{768}{5} - \frac{384}{7}\right) = \frac{3456\pi}{35} \approx 310.21$

b. Revolving about the line $y = 8$, the radius of the disk at x is $8-\sqrt{x^3} = 8-x^{3/2}$.

$V = \pi\int_0^4 (8-x^{3/2})^2\, dx$

$= \pi\int_0^4 (64-16x^{3/2}+x^3)dx$

$= \pi\left[64x - \frac{32}{5}x^{5/2} + \frac{1}{4}x^4\right]_0^4$

$= \pi\left[256 - \frac{1024}{5} + 64\right] = \frac{576\pi}{5} \approx 361.91$

35. The area of a quarter circle with radius 2 is $\int_0^2 \sqrt{4-y^2}\, dy = \pi$.

$\int_0^2 \left[2\sqrt{4-y^2} + 4 - y^2\right]dy$

$= 2\int_0^2 \sqrt{4-y^2}\, dy + \int_0^2 (4-y^2)dy$

$= 2\pi + \left[4y - \frac{1}{3}y^3\right]_0^2 = 2\pi + \left(8 - \frac{8}{3}\right)$

$= 2\pi + \frac{16}{3} \approx 11.62$

36. Let the x-axis lie along the diameter at the base perpendicular to the water level and slice perpendicular to the x-axis. Let $x = 0$ be at the center. The slice has base length $2\sqrt{r^2-x^2}$ and height $\frac{hx}{r}$.

$V = \frac{2h}{r}\int_0^r x\sqrt{r^2-x^2}\,dx$

$= \frac{2h}{r}\left[-\frac{1}{3}\left(r^2-x^2\right)^{3/2}\right]_0^r = \frac{2h}{r}\left(\frac{1}{3}r^3\right) = \frac{2}{3}r^2h$

37. Let the x-axis lie on the base perpendicular to the diameter through the center of the base. The slice at x is a rectangle with base of length $2\sqrt{r^2-x^2}$ and height $x\tan\theta$.

$V = \int_0^r 2x\tan\theta\sqrt{r^2-x^2}\,dx$

$= \left[-\frac{2}{3}\tan\theta(r^2-x^2)^{3/2}\right]_0^r$

$= \frac{2}{3}r^3\tan\theta$

38. **a.** $x = \sqrt[4]{\frac{y}{k}}$

Slice horizontally.

$$\Delta V \approx \pi\left(\sqrt[4]{\frac{y}{k}}\right)^2 \Delta y = \pi\left(\sqrt{\frac{y}{k}}\right)\Delta y$$

If the depth of the tank is h, then

$$V = \pi\int_0^h \sqrt{\frac{y}{k}}dy = \frac{\pi}{\sqrt{k}}\left[\frac{2}{3}y^{3/2}\right]_0^h$$

$$= \frac{2\pi}{3\sqrt{k}}h^{3/2}.$$

The volume as a function of the depth of the tank is $V(y) = \frac{2\pi}{3\sqrt{k}}y^{3/2}$

b. It is given that $\frac{dV}{dt} = -m\sqrt{y}$.

From part **a**, $\frac{dV}{dt} = \frac{\pi}{\sqrt{k}}y^{1/2}\frac{dy}{dt}$.

Thus, $\frac{\pi}{\sqrt{k}}\sqrt{y}\frac{dy}{dt} = -m\sqrt{y}$ and $\frac{dy}{dt} = \frac{-m\sqrt{k}}{\pi}$ which is constant.

39. Let A lie on the xy-plane. Suppose $\Delta A = f(x)\Delta x$ where $f(x)$ is the length at x, so $A = \int f(x)dx$.
Slice the general cone at height z parallel to A. The slice of the resulting region is A_z and ΔA_z is a region related to $f(x)$ and Δx by similar triangles:

$$\Delta A_z = \left(1-\frac{z}{h}\right)f(x)\cdot\left(1-\frac{z}{h}\right)\Delta x$$

$$= \left(1-\frac{z}{h}\right)^2 f(x)\Delta x$$

Therefore, $A_z = \left(1-\frac{z}{h}\right)^2\int f(x)dx = \left(1-\frac{z}{h}\right)^2 A.$

$$\Delta V \approx A_z\Delta z = A\left(1-\frac{z}{h}\right)^2\Delta z \quad V = A\int_0^h\left(1-\frac{z}{h}\right)^2 dz$$

$$= A\left[-\frac{h}{3}\left(1-\frac{z}{h}\right)^3\right]_0^h = \frac{1}{3}Ah.$$

a. $A = \pi r^2$

$$V = \frac{1}{3}Ah = \frac{1}{3}\pi r^2 h$$

b. A face of a regular tetrahedron is an equilateral triangle. If the side of an equilateral triangle has length r, then the area is $A = \frac{1}{2}r\cdot\frac{\sqrt{3}}{2}r = \frac{\sqrt{3}}{4}r^2$.
The center of an equilateral triangle is $\frac{2}{3}\cdot\frac{\sqrt{3}}{2}r = \frac{1}{\sqrt{3}}r$ from a vertex. Then the height of a regular tetrahedron is

$$h = \sqrt{r^2 - \left(\frac{1}{\sqrt{3}}r\right)^2} = \sqrt{\frac{2}{3}r^2} = \frac{\sqrt{2}}{\sqrt{3}}r.$$

$$V = \frac{1}{3}Ah = \frac{\sqrt{2}}{12}r^3$$

40. If two solids have the same cross sectional area at every x in $[a, b]$, then they have the same volume.

41. First we examine the cross-sectional areas of each shape.
Hemisphere: cross-sectional shape is a circle. The radius of the circle at height y is $\sqrt{r^2 - y^2}$.
Therefore, the cross-sectional area for the hemisphere is

$$A_h = \pi(\sqrt{r^2 - y^2})^2 = \pi(r^2 - y^2)$$

Cylinder w/o cone: cross-sectional shape is a washer. The outer radius is a constant , r. The inner radius at height y is equal to y. Therefore, the cross-sectional area is

$$A_2 = \pi r^2 - \pi y^2 = \pi(r^2 - y^2).$$

Since both cross-sectional areas are the same, we can apply Cavaleri's Principle. The volume of the hemisphere of radius r is

$V =$ vol. of cylinder - vol. of cone

$$= \pi r^2 h - \frac{1}{3}\pi r^2 h$$

$$= \frac{2}{3}\pi r^2 h$$

With the height of the cylinder and cone equal to r, the volume of the hemisphere is

$$V = \frac{2}{3}\pi r^2(r) = \frac{2}{3}\pi r^3$$

5.3 Concepts Review

1. $2\pi x\, f(x)\Delta x$

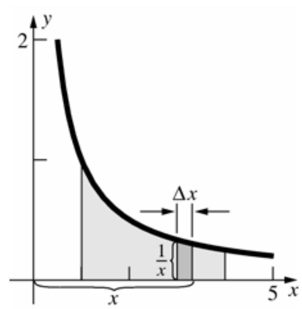
2. $2\pi\int_0^2 x^2 dx;\ \pi\int_0^2 (4-y^2)dy$

3. $2\pi\int_0^2 (1+x)x\, dx$

4. $2\pi\int_0^2 (1+y)(2-y)dy$

Problem Set 5.3

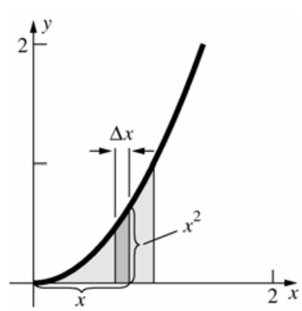
1. **a, b.**



c. $\Delta V \approx 2\pi x\left(\frac{1}{x}\right)\Delta x = 2\pi\Delta x$

d,e. $V = 2\pi\int_1^4 dx = 2\pi\left[x\right]_1^4 = 6\pi \approx 18.85$

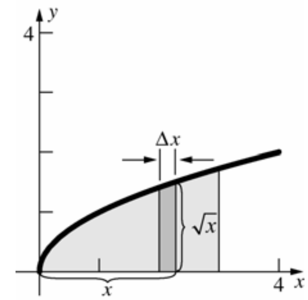
2. **a, b.**



c. $\Delta V \approx 2\pi x(x^2)\Delta x = 2\pi x^3\Delta x$

d, e. $V = 2\pi\int_0^1 x^3 dx = 2\pi\left[\frac{1}{4}x^4\right]_0^1 = \frac{\pi}{2} \approx 1.57$

3. **a, b.**

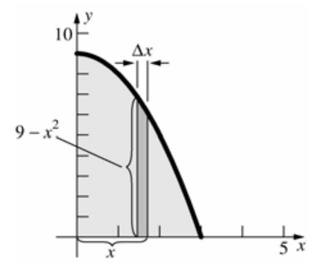


c. $\Delta V \approx 2\pi x\sqrt{x}\,\Delta x = 2\pi x^{3/2}\Delta x$

d, e. $V = 2\pi\int_0^3 x^{3/2}dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^3$

$= \frac{36\sqrt{3}}{5}\pi \approx 39.18$

4. **a,b.**

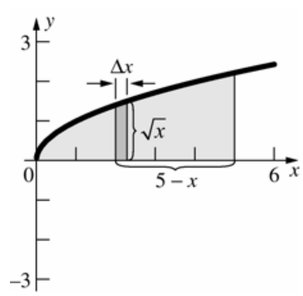


c. $\Delta V \approx 2\pi x(9-x^2)\Delta x = 2\pi(9x-x^3)\Delta x$

d, e. $V = 2\pi\int_0^3 (9x-x^3)dx = 2\pi\left[\frac{9}{2}x^2 - \frac{1}{4}x^4\right]_0^3$

$= 2\pi\left(\frac{81}{2} - \frac{81}{4}\right) = \frac{81\pi}{2} \approx 127.23$

5. **a, b.**



c. $\Delta V \approx 2\pi(5-x)\sqrt{x}\,\Delta x$

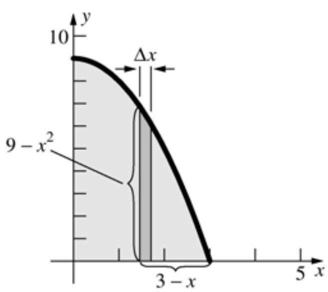
$= 2\pi(5x^{1/2} - x^{3/2})\Delta x$

d, e. $V = 2\pi\int_0^5 (5x^{1/2} - x^{3/2})dx$

$= 2\pi\left[\frac{10}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^5$

$= 2\pi\left(\frac{50\sqrt{5}}{3} - 10\sqrt{5}\right) = \frac{40\sqrt{5}}{3}\pi \approx 93.66$

6. a, b.



c. $\Delta V \approx 2\pi(3-x)(9-x^2)\Delta x$

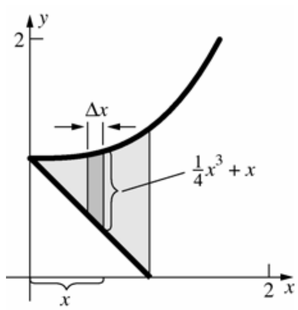
$= 2\pi(27-9x-3x^2+x^3)\Delta x$

d, e. $V = 2\pi\int_0^3 (27-9x-3x^2+x^3)dx$

$= 2\pi\left[27x - \frac{9}{2}x^2 - x^3 + \frac{1}{4}x^4\right]_0^3$

$= 2\pi\left(81 - \frac{81}{2} - 27 + \frac{81}{4}\right) = \frac{135\pi}{2} \approx 212.06$

7. a, b.



c. $\Delta V \approx 2\pi x\left[\left(\frac{1}{4}x^3+1\right)-(1-x)\right]\Delta x$

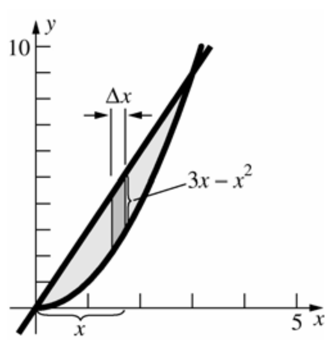
$= 2\pi\left(\frac{1}{4}x^4 + x^2\right)\Delta x$

d, e. $V = 2\pi\int_0^1 \left(\frac{1}{4}x^4 + x^2\right)dx$

$= 2\pi\left[\frac{1}{20}x^5 + \frac{1}{3}x^3\right]_0^1 = 2\pi\left(\frac{1}{20}+\frac{1}{3}\right)$

$= \frac{23\pi}{30} \approx 2.41$

8. a, b.

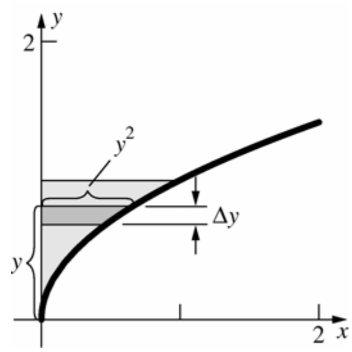


c. $\Delta V \approx 2\pi x(3x - x^2)\Delta x = 2\pi(3x^2 - x^3)\Delta x$

d, e. $V = 2\pi\int_0^3 (3x^2 - x^3)dx = 2\pi\left[x^3 - \frac{1}{4}x^4\right]_0^3$

$= 2\pi\left(27 - \frac{81}{4}\right) = \frac{27\pi}{2} \approx 42.41$

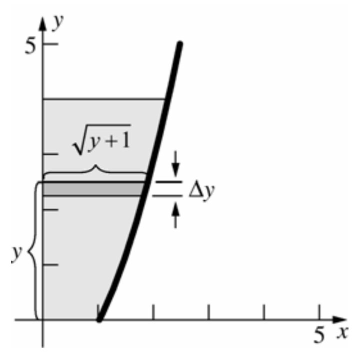
9. a, b.



c. $\Delta V \approx 2\pi y(y^2)\Delta y = 2\pi y^3 \Delta y$

d, e. $V = 2\pi\int_0^1 y^3 dy = 2\pi\left[\frac{1}{4}y^4\right]_0^1 = \frac{\pi}{2} \approx 1.57$

10. a, b.



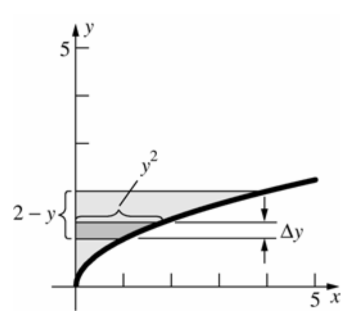
c. $\Delta V \approx 2\pi y\left(\sqrt{y}+1\right)\Delta y = 2\pi(y^{3/2}+y)\Delta y$

d, e. $V = 2\pi\int_0^4 (y^{3/2}+y)\,dy$

$= 2\pi\left[\frac{2}{5}y^{5/2} + \frac{1}{2}y^2\right]_0^4 = 2\pi\left(\frac{64}{5}+8\right)$

$= \frac{208\pi}{5} \approx 130.69$

11. a, b.

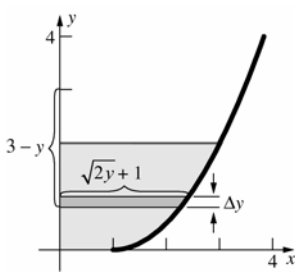


c. $\Delta V \approx 2\pi(2-y)y^2\Delta y = 2\pi(2y^2 - y^3)\Delta y$

d, e. $V = 2\pi\int_0^2 (2y^2 - y^3)dy = 2\pi\left[\frac{2}{3}y^3 - \frac{1}{4}y^4\right]_0^2$

$= 2\pi\left(\frac{16}{3} - 4\right) = \frac{8\pi}{3} \approx 8.38$

12. a, b.



c. $\Delta V \approx 2\pi(3-y)\left(\sqrt{2y}+1\right)\Delta y$

$= 2\pi\left(3 + 3\sqrt{2}y^{1/2} - y - \sqrt{2}y^{3/2}\right)\Delta y$

d, e. $V = 2\pi\int_0^2 \left(3 + 3\sqrt{2}y^{1/2} - y - \sqrt{2}\,y^{3/2}\right)dy$

$= 2\pi\left[3y + 2\sqrt{2}y^{3/2} - \frac{1}{2}y^2 - \frac{2\sqrt{2}}{5}y^{5/2}\right]_0^2$

$= 2\pi\left(6 + 8 - 2 - \frac{16}{5}\right) = \frac{88\pi}{5} \approx 55.29$

13. a. $\pi\int_a^b \left[f(x)^2 - g(x)^2\right]dx$

b. $2\pi\int_a^b x\left[f(x) - g(x)\right]dx$

c. $2\pi\int_a^b (x-a)\left[f(x) - g(x)\right]dx$

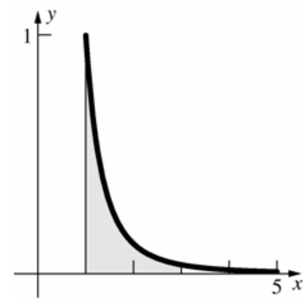
d. $2\pi\int_a^b (b-x)\left[f(x) - g(x)\right]dx$

14. a. $\pi\int_c^d \left[f(y)^2 - g(y)^2\right]dy$

b. $2\pi\int_c^d y\left[f(y) - g(y)\right]dy$

c. $2\pi\int_c^d (3-y)\left[f(y) - g(y)\right]dy$

15.



a. $A = \int_1^3 \frac{1}{x^3}dx$

b. $V = 2\pi\int_1^3 x\left(\frac{1}{x^3}\right)dx = 2\pi\int_1^3 \frac{1}{x^2}dx$

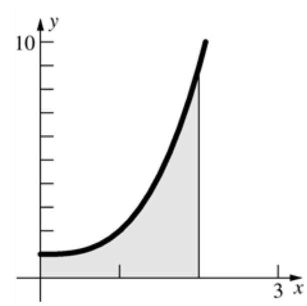
c. $V = \pi\int_1^3 \left[\left(\frac{1}{x^3} + 1\right)^2 - (-1)^2\right]dx$

$= \pi\int_1^3 \left(\frac{1}{x^6} + \frac{2}{x^3}\right)dx$

d. $V = 2\pi\int_1^3 (4-x)\left(\frac{1}{x^3}\right)dx$

$= 2\pi\int_1^3 \left(\frac{4}{x^3} - \frac{1}{x^2}\right)dx$

16.



a. $A = \int_0^2 (x^3 + 1)\,dx$

b. $V = 2\pi\int_0^2 x(x^3 + 1)dx = 2\pi\int_0^2 (x^4 + x)dx$

c. $V = \pi\int_0^2 \left[(x^3 + 2)^2 - (-1)^2\right]$

$= \pi\int_0^2 (x^6 + 4x^3 + 3)dx$

d. $V = 2\pi\int_0^2 (4-x)(x^3 + 1)dx$

$= 2\pi\int_0^2 (-x^4 + 4x^3 - x + 4)dx$

17. To find the intersection point, solve $\sqrt{y} = \frac{y^3}{32}$.

$$y = \frac{y^6}{1024}$$
$$y^6 - 1024y = 0$$
$$y(y^5 - 1024) = 0$$
$y = 0, 4$
$$V = 2\pi\int_0^4 y\left(\sqrt{y} - \frac{y^3}{32}\right)dy$$
$$= 2\pi\int_0^4 \left(y^{3/2} - \frac{y^4}{32}\right)dy$$
$$= 2\pi\left[\frac{2}{5}y^{5/2} - \frac{y^5}{160}\right]_0^4 = 2\pi\left(\frac{64}{5} - \frac{32}{5}\right) = \frac{64\pi}{5}$$
≈ 40.21

18. $$V = 2\pi\int_0^4 (4-y)\left(\sqrt{y} - \frac{y^3}{32}\right)dy$$
$$= 2\pi\int_0^4 \left(4y^{1/2} - y^{3/2} - \frac{y^3}{8} + \frac{y^4}{32}\right)dy$$
$$= 2\pi\left[\frac{8}{3}y^{3/2} - \frac{2}{5}y^{5/2} - \frac{y^4}{32} + \frac{y^5}{160}\right]_0^4$$
$$= 2\pi\left(\frac{64}{3} - \frac{64}{5} - 8 + \frac{32}{5}\right) = \frac{208\pi}{15} \approx 43.56$$

19. Let R be the region bounded by $y = \sqrt{b^2 - x^2}$, $y = -\sqrt{b^2 - x^2}$, and $x = a$. When R is revolved about the y-axis, it produces the desired solid.
$$V = 2\pi\int_a^b x\left(\sqrt{b^2 - x^2} + \sqrt{b^2 - x^2}\right)dx$$
$$= 4\pi\int_a^b x\sqrt{b^2 - x^2}\, dx = 4\pi\left[-\frac{1}{3}(b^2 - x^2)^{3/2}\right]_a^b$$
$$= 4\pi\left[\frac{1}{3}(b^2 - a^2)^{3/2}\right] = \frac{4\pi}{3}(b^2 - a^2)^{3/2}$$

20. $y = \pm\sqrt{a^2 - x^2}, -a \le x \le a$
$$V = 2\pi\int_{-a}^a (b-x)\left(2\sqrt{a^2 - x^2}\right)dx$$
$$= 4\pi b\int_{-a}^a \sqrt{a^2 - x^2}\, dx - 4\pi\int_{-a}^a x\sqrt{a^2 - x^2}\, dx$$
$$= 4\pi b\left(\frac{1}{2}\pi a^2\right) - 4\pi\left[-\frac{1}{3}(a^2 - x^2)^{3/2}\right]_{-a}^a = 2\pi^2 a^2 b$$
(Note that the area of a semicircle with radius a is $\int_{-a}^a \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\pi a^2$.)

21. To find the intersection point, solve $\sin(x^2) = \cos(x^2)$.
$$\tan(x^2) = 1$$
$$x^2 = \frac{\pi}{4}$$
$$x = \frac{\sqrt{\pi}}{2}$$
$$V = 2\pi\int_0^{\sqrt{\pi}/2} x\left[\cos(x^2) - \sin(x^2)\right]dx$$
$$= 2\pi\int_0^{\sqrt{\pi}/2} \left[x\cos(x^2) - x\sin(x^2)\right]dx$$
$$= 2\pi\left[\frac{1}{2}\sin(x^2) + \frac{1}{2}\cos(x^2)\right]_0^{\sqrt{\pi}/2}$$
$$= 2\pi\left[\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right) - \frac{1}{2}\right] = \pi\left(\sqrt{2} - 1\right) \approx 1.30$$

22. $$V = 2\pi\int_0^{2\pi} x(2 + \sin x)dx$$
$$= 2\pi\int_0^{2\pi} (2x + x\sin x)dx$$
$$= 2\pi\int_0^{2\pi} 2x\, dx + 2\pi\int_0^{2\pi} x\sin x\, dx$$
$$= 2\pi\left[x^2\right]_0^{2\pi} + 2\pi\left[\sin x - x\cos x\right]_0^{2\pi}$$
$$= 2\pi(4\pi^2) + 2\pi(-2\pi) = 4\pi^2(2\pi - 1) \approx 208.57$$

23. **a.** The curves intersect when $x = 0$ and $x = 1$.
$$V = \pi\int_0^1 [x^2 - (x^2)^2]dx = \pi\int_0^1 (x^2 - x^4)dx$$
$$= \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15} \approx 0.42$$

b. $$V = 2\pi\int_0^1 x(x - x^2)dx = 2\pi\int_0^1 (x^2 - x^3)dx$$
$$= 2\pi\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$$
≈ 0.52

c. Slice perpendicular to the line $y = x$. At (a, a), the perpendicular line has equation $y = -(x-a) + a = -x + 2a$. Substitute $y = -x + 2a$ into $y = x^2$ and solve for $x \ge 0$.
$$x^2 + x - 2a = 0$$
$$x = \frac{-1 \pm \sqrt{1+8a}}{2}$$
$$x = \frac{-1 + \sqrt{1+8a}}{2}$$

Substitute into $y = -x + 2a$, so

$y = \frac{1+4a-\sqrt{1+8a}}{2}$. Find an expression for r^2, the square of the distance from (a, a) to $\left(\frac{-1+\sqrt{1+8a}}{2}, \frac{1+4a-\sqrt{1+8a}}{2}\right)$.

$$r^2 = \left[a - \frac{-1+\sqrt{1+8a}}{2}\right]^2 + \left[a - \frac{1+4a-\sqrt{1+8a}}{2}\right]^2$$

$$= \left[\frac{2a+1-\sqrt{1+8a}}{2}\right]^2 + \left[-\frac{2a+1-\sqrt{1+8a}}{2}\right]^2$$

$$= 2\left[\frac{2a+1-\sqrt{1+8a}}{2}\right]^2$$

$$= 2a^2 + 6a + 1 - 2a\sqrt{1+8a} - \sqrt{1+8a}$$

$$\Delta V \approx \pi r^2 \Delta a$$

$$V = \pi\int_0^1 (2a^2 + 6a + 1 - 2a\sqrt{1+8a} - \sqrt{1+8a})\, da$$

$$= \pi\left[\frac{2}{3}a^3 + 3a^2 + a - \frac{1}{12}(1+8a)^{3/2}\right]_0^1 - \pi\int_0^1 2a\sqrt{1+8a}\, da$$

$$= \pi\left[\left(\frac{2}{3} + 3 + 1 - \frac{9}{4}\right) - \left(-\frac{1}{12}\right)\right] - \pi\int_0^1 2a\sqrt{1+8a}\, da$$

$$= \frac{5\pi}{2} - \pi\int_0^1 2a\sqrt{1+8a}\, da$$

To integrate $\int_0^1 2a\sqrt{1+8a}\, da$, use the substitution $u = 1 + 8a$.

$$\int_0^1 2a\sqrt{1+8a}\, da = \int_1^9 \frac{1}{4}(u-1)\sqrt{u}\,\frac{1}{8}du$$

$$= \frac{1}{32}\int_1^9 (u^{3/2} - u^{1/2})du$$

$$= \frac{1}{32}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^9$$

$$= \frac{1}{32}\left[\left(\frac{486}{5} - 18\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right] = \frac{149}{60}$$

$$V = \frac{5\pi}{2} - \frac{149\pi}{60} = \frac{\pi}{60} \approx 0.052$$

24. $\Delta V \approx 4\pi x^2 \Delta x$

$$V = 4\pi\int_0^r x^2\, dx = 4\pi\left[\frac{1}{3}x^3\right]_0^r = \frac{4}{3}\pi r^3$$

25. $\Delta V \approx \frac{x^2}{r^2} S\Delta x$

$$V = \frac{S}{r^2}\int_0^r x^2 dx = \frac{S}{r^2}\left[\frac{1}{3}x^3\right]_0^r = \frac{1}{3}rS$$

5.4 Concepts Review

1. Circle

$x^2 + y^2 = 16\cos^2 t + 16\sin^2 t = 16$

2. x; $x^2 + 1$

3. $\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$

4. Mean Value Theorem (for derivatives)

Problem Set 5.4

1. $f(x) = 4x^{3/2}$, $f'(x) = 6x^{1/2}$

$$L = \int_{1/3}^5 \sqrt{1 + (6x^{1/2})^2}\, dx = \int_{1/3}^5 \sqrt{1+36x}\, dx$$

$$= \left[\frac{1}{36}\cdot\frac{2}{3}(1+36x)^{3/2}\right]_{1/3}^5$$

$$= \frac{1}{54}\left(181\sqrt{181} - 13\sqrt{13}\right) \approx 44.23$$

2. $f(x) = \frac{2}{3}(x^2+1)^{3/2}$, $f'(x) = 2x(x^2+1)^{1/2}$

$$L = \int_1^2 \sqrt{1 + \left[2x(x^2+1)^{1/2}\right]^2}\, dx$$

$$= \int_1^2 \sqrt{4x^4 + 4x^2 + 1}\, dx = \int_1^2 (2x^2+1)dx$$

$$= \left[\frac{2}{3}x^3 + x\right]_1^2 = \left(\frac{16}{3} + 2\right) - \left(\frac{2}{3} + 1\right) = \frac{17}{3} \approx 5.67$$

3. $f(x)=(4-x^{2/3})^{3/2}$,

$$f'(x)=\frac{3}{2}(4-x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right)$$
$$=-x^{-1/3}(4-x^{2/3})^{1/2}$$
$$L=\int_1^8\sqrt{1+\left[-x^{-1/3}(4-x^{2/3})^{1/2}\right]^2}\,dx$$
$$=\int_1^8\sqrt{4x^{-2/3}}\,dx=\int_1^8 2x^{-1/3}dx$$
$$=2\left[\frac{3}{2}x^{2/3}\right]_1^8=3(4-1)=9$$

4. $f(x)=\dfrac{x^4+3}{6x}=\dfrac{x^3}{6}+\dfrac{1}{2x}$

$$f'(x)=\frac{x^2}{2}-\frac{1}{2x^2}$$
$$L=\int_1^3\sqrt{1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2}\,dx$$
$$=\int_1^3\sqrt{\frac{x^4}{4}+\frac{1}{2}+\frac{1}{4x^4}}\,dx=\int_1^3\sqrt{\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)^2}\,dx$$
$$=\int_1^3\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)dx=\left[\frac{x^3}{6}-\frac{1}{2x}\right]_1^3$$
$$=\left(\frac{9}{2}-\frac{1}{6}\right)-\left(\frac{1}{6}-\frac{1}{2}\right)=\frac{14}{3}\approx 4.67$$

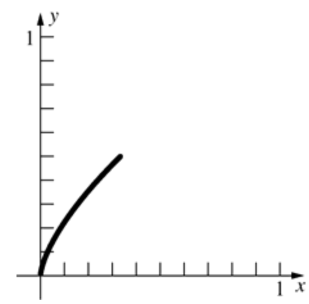
5. $g(y)=\dfrac{y^4}{16}+\dfrac{1}{2y^2},\ g'(y)=\dfrac{y^3}{4}-\dfrac{1}{y^3}$

$$L=\int_{-3}^{-2}\sqrt{1+\left(\frac{y^3}{4}-\frac{1}{y^3}\right)^2}\,dy$$
$$=\int_{-3}^{-2}\sqrt{\frac{y^6}{16}+\frac{1}{2}+\frac{1}{y^6}}\,dy=\int_{-3}^{-2}\sqrt{\left(\frac{y^3}{4}+\frac{1}{y^3}\right)^2}\,dy$$
$$=\int_{-3}^{-2}-\left(\frac{y^3}{4}+\frac{1}{y^3}\right)dy=-\left[\frac{y^4}{16}-\frac{1}{2y^2}\right]_{-3}^{-2}$$
$$=-\left[\left(1-\frac{1}{8}\right)-\left(\frac{81}{16}-\frac{1}{18}\right)\right]=\frac{595}{144}\approx 4.13$$

6. $x=\dfrac{y^5}{30}+\dfrac{1}{2y^3}$

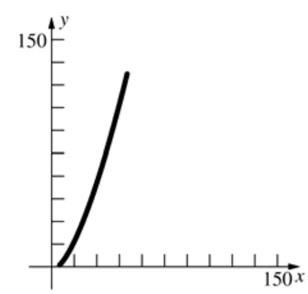
$$g(y)=\frac{y^5}{30}+\frac{1}{2y^3},\ g'(y)=\frac{y^4}{6}-\frac{3}{2y^4}$$
$$L=\int_1^3\sqrt{1+\left(\frac{y^4}{6}-\frac{3}{2y^4}\right)^2}\,dy$$
$$=\int_1^3\sqrt{\frac{y^8}{36}+\frac{1}{2}+\frac{9}{4y^8}}\,dy=\int_1^3\sqrt{\left(\frac{y^4}{6}+\frac{3}{2y^4}\right)^2}\,dy$$
$$=\int_1^3\left(\frac{y^4}{6}+\frac{3}{2y^4}\right)dy=\left[\frac{y^5}{30}-\frac{1}{2y^3}\right]_1^3$$
$$=\left(\frac{81}{10}-\frac{1}{54}\right)-\left(\frac{1}{30}-\frac{1}{2}\right)=\frac{1154}{135}\approx 8.55$$

7.



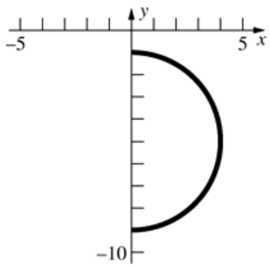
$$\frac{dx}{dt}=t^2,\ \frac{dy}{dt}=t$$
$$L=\int_0^1\sqrt{(t^2)^2+(t)^2}\,dt=\int_0^1\sqrt{t^4+t^2}\,dt$$
$$=\int_0^1 t\sqrt{t^2+1}\,dt=\left[\frac{1}{3}(t^2+1)^{3/2}\right]_0^1=\frac{1}{3}\left(2\sqrt{2}-1\right)$$
$$\approx 0.61$$

8.



$$\frac{dx}{dt}=6t,\ \frac{dy}{dt}=6t^2$$
$$L=\int_1^4\sqrt{(6t)^2+(6t^2)^2}\,dt=\int_1^4\sqrt{36t^2+36t^4}\,dt$$
$$=\int_1^4 6t\sqrt{1+t^2}\,dt=\left[2(1+t^2)^{3/2}\right]_1^4$$
$$=2\left(17\sqrt{17}-2\sqrt{2}\right)\approx 134.53$$

9.



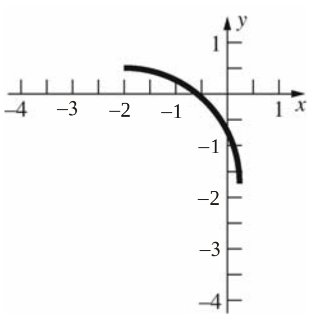
$$\frac{dx}{dt} = 4\cos t, \frac{dy}{dt} = -4\sin t$$

$$L = \int_0^{\pi} \sqrt{(4\cos t)^2 + (-4\sin t)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{16\cos^2 t + 16\sin^2 t}\, dt = \int_0^{\pi} 4dt$$

$$= 4\pi \approx 12.57$$

10.



$$\frac{dx}{dt} = 2\sqrt{5}\cos 2t, \frac{dy}{dt} = -2\sqrt{5}\sin 2t$$

$$L = \int_0^{\pi/4} \sqrt{\left(2\sqrt{5}\cos 2t\right)^2 + \left(-2\sqrt{5}\sin 2t\right)^2}\, dt$$

$$= \int_0^{\pi/4} \sqrt{20\cos^2 2t + 20\sin^2 2t}\, dt = \int_0^{\pi/4} 2\sqrt{5}\, dt$$

$$= \frac{\sqrt{5}\,\pi}{2} \approx 3.51$$

11. $f(x) = 2x + 3,\ f'(x) = 2$

$$L = \int_1^3 \sqrt{1 + (2)^2}\, dx = \sqrt{5}\int_1^3 dx = 2\sqrt{5}$$

At $x = 1$, $y = 2(1) + 3 = 5$.
At $x = 3$, $y = 2(3) + 3 = 9$.

$$d = \sqrt{(3-1)^2 + (9-5)^2} = \sqrt{20} = 2\sqrt{5}$$

12. $x = y + \frac{3}{2}$

$$g(y) = y + \frac{3}{2}, g'(y) = 1$$

$$L = \int_1^3 \sqrt{1 + (1)^2} = \sqrt{2}\int_1^3 dy = 2\sqrt{2}$$

At $y = 1$, $x = 1 + \frac{3}{2} = \frac{5}{2}$.

At $y = 3$, $x = 3 + \frac{3}{2} = \frac{9}{2}$.

$$d = \sqrt{\left(\frac{9}{2} - \frac{5}{2}\right)^2 + (3-1)^2} = \sqrt{8} = 2\sqrt{2}$$

13. $\frac{dx}{dt} = 1, \frac{dy}{dt} = 2t$

$$L = \int_0^2 \sqrt{1^2 + (2t)^2}\, dt = \int_0^2 \sqrt{1 + 4t^2}\, dt$$

Let $f(t) = \sqrt{1 + 4t^2}$. Using the Parabolic Rule

with n = 8,

$$L \approx \frac{2-0}{3\times 8}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right)\right.$$

$$\left. +2f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + f(2)\right]$$

$$\approx \frac{1}{12}[1 + 4\times 1.118 + 2\times 1.4142$$
$$+ 4\times 1.8028 + 2\times 2.2361$$
$$+ 4\times 2.6926 + 2\times 3.1623$$
$$+ 4\times 3.6401 + 4.1231] \approx 4.6468$$

14. $\frac{dx}{dt} = 2t, \frac{dy}{dt} = \frac{1}{2\sqrt{t}}$

$$L \approx \int_1^4 \sqrt{(2t)^2 + \left(\frac{1}{2\sqrt{t}}\right)^2}\, dt = \int_1^4 \sqrt{4t^2 + \frac{1}{4t}}\, dt$$

Let $f(t) = \sqrt{4t^2 + \frac{1}{4t}}$. Using the Parabolic Rule

with n = 8,

$$L \approx \frac{4-1}{3\times 8}\left[f(1) + 4f\left(\frac{11}{8}\right) + 2f\left(\frac{14}{8}\right)\right.$$

$$+4f\left(\frac{17}{8}\right) + 2f\left(\frac{20}{8}\right) + 4f\left(\frac{23}{8}\right) + 2f\left(\frac{26}{8}\right)$$

$$\left. +4f\left(\frac{29}{8}\right) + f(4)\right] \approx \frac{1}{8}\big(2.0616 + 4\times 2.8118$$

$$+2\times 3.562 + 4\times 4.312 + 2\times 5.0621 + 4\times 5.8122$$

$$2\times 6.5622 + 4\times 7.3122 + 8.0623\big) \approx 15.0467$$

15. $\frac{dx}{dt}=\cos t, \frac{dy}{dt}=-2\sin 2t$

$L=\int_0^{\pi/2}\sqrt{(\cos t)^2+(-2\sin 2t)^2}\,dt$

$=\int_0^{\pi/2}\sqrt{\cos^2 t+4\sin^2 2t}\,dt$

Let $f(t)=\sqrt{\cos^2 t+4\sin^2 2t}$. Using the Parabolic Rule with n = 8,

$L\approx\frac{\pi/2-0}{3\times 8}\left[f(0)+4f\left(\frac{\pi}{16}\right)+2f\left(\frac{2\pi}{16}\right)\right.$

$+4f\left(\frac{3\pi}{16}\right)+2f\left(\frac{4\pi}{16}\right)+4f\left(\frac{5\pi}{16}\right)+2f\left(\frac{6\pi}{16}\right)$

$\left.+4f\left(\frac{7\pi}{16}\right)+f\left(\frac{\pi}{2}\right)\right]\approx\frac{\pi}{48}[1+4\times 1.2441$

$+2\times 1.6892+4\times 2.0262+2\times 2.1213+4\times 1.9295$

$+2\times 1.4651+4\times 0.7898+0)\approx 2.3241$

16. $\frac{dx}{dt}=1, \frac{dy}{dt}=\sec^2 t$

$L=\int_0^{\pi/4}\sqrt{1^2+(\sec^2 t)^2}\,dt=\int_0^{\pi/4}\sqrt{1+\sec^4 t}\,dt$

Let $f(t)=\sqrt{1+\sec^4 t}$. Using the Parabolic Rule with n = 8, $L\approx\frac{\pi/4-0}{3\times 8}\left[f(0)+4f\left(\frac{\pi}{32}\right)\right.$

$+2f\left(\frac{2\pi}{32}\right)+4f\left(\frac{3\pi}{32}\right)+2f\left(\frac{4\pi}{32}\right)+4f\left(\frac{5\pi}{32}\right)$

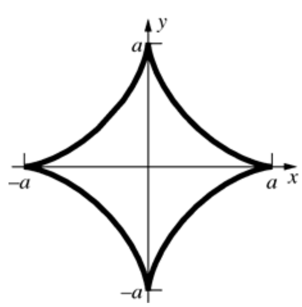
$\left.+2f\left(\frac{6\pi}{32}\right)+4f\left(\frac{7\pi}{32}\right)+f\left(\frac{\pi}{4}\right)\right]$

$\approx\frac{\pi}{96}[1.4142+4\times 1.4211+2\times 1.4425+4\times 1.4807$

$+2\times 1.5403+4\times 1.6288+2\times 1.7585$

$+4\times 1.9495+2.2361]\approx 1.278$

17.



$\frac{dx}{dt}=3a\cos t\sin^2 t, \frac{dy}{dt}=-3a\sin t\cos^2 t$

The first quadrant length is L

$=\int_0^{\pi/2}\sqrt{(3a\cos t\sin^2 t)^2+(-3a\sin t\cos^2 t)^2}\,dt$

$=\int_0^{\pi/2}\sqrt{9a^2\cos^2 t\sin^4 t+9a^2\sin^2 t\cos^4 t}\,dt$

$=\int_0^{\pi/2}\sqrt{9a^2\cos^2 t\sin^2 t(\sin^2 t+\cos^2 t)}\,dt$

$=\int_0^{\pi/2}3a\cos t\sin t\,dt=3a\left[-\frac{1}{2}\cos^2 t\right]_0^{\pi/2}=\frac{3a}{2}$

(The integral can also be evaluated as $3a\left[\frac{1}{2}\sin^2 t\right]_0^{\pi/2}$ with the same result.)

The total length is $6a$.

18. **a.** $\overline{OT}=\text{length}\left(\widehat{PT}\right)=a\theta$

b. From Figure 18 of the text,

$\sin\theta=\frac{\overline{PQ}}{\overline{PC}}=\frac{\overline{PQ}}{a}$ and $\cos\theta=\frac{\overline{QC}}{\overline{PC}}=\frac{\overline{QC}}{a}$.

Therefore $\overline{PQ}=a\sin\theta$ and $\overline{QC}=a\cos\theta$.

c. $x=\overline{OT}-\overline{PQ}=a\theta-a\sin\theta=a(\theta-\sin\theta)$

$y=\overline{CT}-\overline{CQ}=a-a\cos\theta=a(1-\cos\theta)$

19. From Problem 18,

$x=a(\theta-\sin\theta), y=a(1-\cos\theta)$

$\frac{dx}{d\theta}=a(1-\cos\theta), \frac{dy}{d\theta}=a\sin\theta$ so

$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=[a(1-\cos\theta)]^2+[a\sin\theta]^2$

$=a^2-2a^2\cos\theta+a^2\cos^2\theta+a^2\sin^2\theta$

$=2a^2-2a^2\cos\theta=2a^2(1-\cos\theta)$

$=4a^2\frac{1-\cos\theta}{2}=4a^2\sin^2\left(\frac{\theta}{2}\right)$.

The length of one arch of the cycloid is

$\int_0^{2\pi}\sqrt{4a^2\sin^2\left(\frac{\theta}{2}\right)}\,d\theta=\int_0^{2\pi}2a\sin\left(\frac{\theta}{2}\right)d\theta$

$=2a\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi}=2a(2+2)=8a$

20. a. Using $\theta = \omega t$, the point P is at $x = a\omega t - a\sin(\omega t)$, $y = a - a\cos(\omega t)$ at time t.

$$\frac{dx}{dt} = a\omega - a\omega\cos(\omega t) = a\omega(1-\cos(\omega t))$$

$$\frac{dy}{dt} = a\omega\sin(\omega t)$$

$$\frac{ds}{dt} = \sqrt{\left[\frac{dy}{dt}\right]^2 + \left[\frac{dx}{dt}\right]^2}$$

$$= \sqrt{a^2\omega^2\sin^2(\omega t) + a^2\omega^2 - 2a^2\omega^2\cos(\omega t) + a^2\omega^2\cos^2(\omega t)} = \sqrt{2a^2\omega^2 - 2a^2\omega^2\cos(\omega t)}$$

$$= 2a\omega\sqrt{\frac{1}{2}(1-\cos(\omega t))} = 2a\omega\sqrt{\sin^2\frac{\omega t}{2}} = 2a\omega\left|\sin\frac{\omega t}{2}\right|$$

b. The speed is a maximum when $\left|\sin\frac{\omega t}{2}\right| = 1$, which occurs when $t = \frac{\pi}{\omega}(2k+1)$. The speed is a minimum when $\left|\sin\frac{\omega t}{2}\right| = 0$, which occurs when $t = \frac{2k\pi}{\omega}$.

c. From Problem 18a, the distance traveled by the wheel is $a\theta$, so at time t, the wheel has gone $a\theta = a\omega t$ miles. Since the car is going 60 miles per hour, the wheel has gone $60t$ miles at time t. Thus, $a\omega = 60$ and the maximum speed of the bug on the wheel is $2a\omega = 2(60) = 120$ miles per hour.

21. a. $\frac{dy}{dx} = \sqrt{x^3-1}$

$$L = \int_1^2 \sqrt{1+x^3-1}\,dx = \int_1^2 x^{3/2}dx$$

$$= \left[\frac{2}{5}x^{5/2}\right]_1^2 = \frac{2}{5}\left(4\sqrt{2}-1\right) \approx 1.86$$

b. $f'(t) = 1-\cos t,\ g'(t) = \sin t$

$$L = \int_0^{4\pi}\sqrt{2-2\cos t}\,dt = \int_0^{4\pi} 2\left|\sin\left(\frac{t}{2}\right)\right|dt$$

$\sin\left(\frac{t}{2}\right)$ is positive for $0 < t < 2\pi$, and by symmetry, we can double the integral from 0 to 2π.

$$L = 4\int_0^{2\pi}\sin\left(\frac{t}{2}\right)dt = \left[-8\cos\frac{t}{2}\right]_0^{2\pi}$$

$= 8 + 8 = 16$

22. a. $\frac{dy}{dx} = \sqrt{64\sin^2 x\cos^4 x - 1}$

$$L = \int_{\pi/6}^{\pi/3}\sqrt{1+64\sin^2\cos^4 x-1}\,dx$$

$$= \int_{\pi/6}^{\pi/3} 8\sin x\cos^2 x dx = \left[-\frac{8}{3}\cos^3 x\right]_{\pi/6}^{\pi/3}$$

$$= -\frac{1}{3} + \sqrt{3} \approx 1.40$$

b. $\frac{dx}{dt} = -a\sin t + a\sin t + at\cos t = at\cos t$

$$\frac{dy}{dt} = a\cos t - a\cos t + at\sin t = at\sin t$$

$$L = \int_{-1}^{1}\sqrt{a^2t^2\cos^2 t + a^2t^2\sin^2 t}dt$$

$$= \int_{-1}^{1}|at|\,dt = \int_0^1 at\,dt - \int_{-1}^0 at\,dt$$

$$= \left[\frac{a}{2}t^2\right]_0^1 - \left[\frac{a}{2}t^2\right]_{-1}^0 = \frac{a}{2} + \frac{a}{2} = a$$

23. $f(x) = 6x,\ f'(x) = 6$

$$A = 2\pi\int_0^1 6x\sqrt{1+36}\,dx = 12\sqrt{37}\pi\int_0^1 x\,dx$$

$$= 12\sqrt{37}\pi\left[\frac{1}{2}x^2\right]_0^1 = 6\sqrt{37}\pi \approx 114.66$$

24. $f(x) = \sqrt{25-x^2},\ f'(x) = -\frac{x}{\sqrt{25-x^2}}$

$$A = 2\pi\int_{-2}^{3}\sqrt{25-x^2}\sqrt{1+\frac{x^2}{25-x^2}}dx$$

$$= 2\pi\int_{-2}^{3}\sqrt{25-x^2+x^2}\,dx$$

$$= 2\pi\int_{-2}^{3}5dx = 10\pi[x]_{-2}^{3} = 50\pi \approx 157.08$$

25. $f(x)=\frac{x^3}{3},\ f'(x)=x^2$

$$A=2\pi\int_1^{\sqrt{7}}\frac{x^3}{3}\sqrt{1+x^4}\,dx$$

$$=2\pi\left[\frac{1}{18}(1+x^4)^{3/2}\right]_1^{\sqrt{7}}=\frac{\pi}{9}\left(250\sqrt{2}-2\sqrt{2}\right)$$

$$=\frac{248\pi\sqrt{2}}{9}\approx 122.43$$

26. $f(x)=\frac{x^6+2}{8x^2}=\frac{x^4}{8}+\frac{1}{4x^2},\ f'(x)=\frac{x^3}{2}-\frac{1}{2x^3}$

$$A=2\pi\int_1^3\left(\frac{x^4}{8}+\frac{1}{4x^2}\right)\sqrt{1+\left(\frac{x^3}{2}-\frac{1}{2x^3}\right)^2}$$

$$=2\pi\int_1^3\left(\frac{x^4}{8}+\frac{1}{4x^2}\right)\sqrt{\frac{x^6}{4}+\frac{1}{2}+\frac{1}{4x^6}}\,dx$$

$$=2\pi\int_1^3\left(\frac{x^4}{8}+\frac{1}{4x^2}\right)\left(\frac{x^3}{2}+\frac{1}{2x^3}\right)dx$$

$$=2\pi\int_1^3\left(\frac{x^7}{16}+\frac{3x}{16}+\frac{1}{8x^5}\right)dx$$

$$=2\pi\left[\frac{x^8}{128}+\frac{3x^2}{32}-\frac{1}{32x^4}\right]_1^3$$

$$=2\pi\left[\left(\frac{6561}{128}+\frac{27}{32}-\frac{1}{2592}\right)-\left(\frac{1}{128}+\frac{3}{32}-\frac{1}{32}\right)\right]$$

$$=\frac{8429\pi}{81}\approx 326.92$$

27. $\frac{dx}{dt}=1,\frac{dy}{dt}=3t^2$

$$A=2\pi\int_0^1 t^3\sqrt{1+9t^4}\,dt$$

$$=2\pi\left[\frac{1}{54}(1+9t^4)^{3/2}\right]_0^1=\frac{\pi}{27}\left(10\sqrt{10}-1\right)$$

$$\approx 3.56$$

28. $\frac{dx}{dt}=-2t,\frac{dy}{dt}=2$

$$A=2\pi\int_0^1 2t\sqrt{4t^2+4}\,dt=8\pi\int_0^1 t\sqrt{t^2+1}\,dt$$

$$=8\pi\left[\frac{1}{3}(t^2+1)^{3/2}\right]_0^1=\frac{8\pi}{3}(2\sqrt{2}-1)\approx 15.32$$

29. $y=f(x)=\sqrt{r^2-x^2}$

$f'(x)=-x(r^2-x^2)^{-1/2}$

$$A=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\left[-x(r^2-x^2)^{-1/2}\right]^2}\,dx$$

$$=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+x^2(r^2-x^2)^{-1}}\,dx$$

$$=2\pi\int_{-r}^{r}\sqrt{\left(r^2-x^2\right)\left(1+x^2(r^2-x^2)^{-1}\right)}\,dx$$

$$=2\pi\int_{-r}^{r}\sqrt{r^2-x^2+x^2}\,dx$$

$$=2\pi\int_{-r}^{r}\sqrt{r^2}\,dx=2\pi\int_{-r}^{r}r\,dx=2\pi rx\,\Big|_{-r}^{r}=4\pi r^2$$

30. $x=f(t)=r\cos t$

$y=g(t)=r\sin t$

$f'(t)=-r\sin t$

$g'(t)=r\cos t$

$$A=2\pi\int_0^{\pi}r\sin t\sqrt{(-r\sin t)^2+(r\cos t)^2}\,dt$$

$$=2\pi\int_0^{\pi}r\sin t\sqrt{r^2\sin^2 t+r^2\cos^2 t}\,dt$$

$$=2\pi\int_0^{\pi}r\sin t\sqrt{r^2}\,dt$$

$$=2\pi\int_0^{\pi}r^2\sin t\,dt=-2\pi r^2\cos t\,\Big|_0^{\pi}$$

$$=-2r^2(-1-1)=4\pi r^2$$

31. a. The base circumference is equal to the arc length of the sector, so $2\pi r=\theta l.$ Therefore, $\theta=\frac{2\pi r}{l}$.

b. The area of the sector is equal to the lateral surface area. Therefore, the lateral surface area is $\frac{1}{2}l^2\theta=\frac{1}{2}l^2\left(\frac{2\pi r}{l}\right)=\pi rl$.

c. Assume $r_2>r_1$. Let l_1 and l_2 be the slant heights for r_1 and r_2, respectively. Then $A=\pi r_2 l_2-\pi r_1 l_1=\pi r_2(l_1+l)-\pi r_1 l_1$.

From part a, $\theta=\frac{2\pi r_2}{l_2}=\frac{2\pi r_2}{l_1+l}=\frac{2\pi r_1}{l_1}$.

Solve for $l_1: l_1 r_2=l_1 r_1+l r_1$

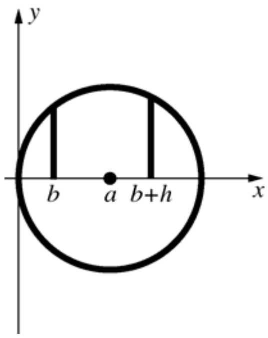
$l_1(r_2-r_1)=l r_1$

$$l_1=\frac{l r_1}{r_2-r_1}$$

$$A=\pi r_2\left(\frac{l r_1}{r_2-r_1}+l\right)-\pi r_1\left(\frac{l r_1}{r_2-r_1}\right)$$

$$=\pi(l r_1+l r_2)=2\pi\left[\frac{r_1+r_2}{2}\right]l$$

32. Put the center of a circle of radius a at $(a, 0)$. Revolving the portion of the circle from $x = b$ to $x = b + h$ about the x-axis results in the surface in question. (See figure.)



The equation of the top half of the circle is

$y = \sqrt{a^2 - (x-a)^2}$.

$$\frac{dy}{dx} = \frac{-(x-a)}{\sqrt{a^2-(x-a)^2}}$$

$$A = 2\pi\int_b^{b+h} \sqrt{a^2-(x-a)^2}\sqrt{1+\frac{(x-a)^2}{a^2-(x-a)^2}}\,dx$$

$$= 2\pi\int_b^{b+h} \sqrt{a^2-(x-a)^2+(x-a)^2}\,dx$$

$$= 2\pi\int_b^{b+h} a\,dx = 2\pi a[x]_b^{b+h} = 2\pi ah$$

A right circular cylinder of radius a and height h has surface area $2\pi ah$.

33. a. $\frac{dx}{dt} = a(1-\cos t), \frac{dy}{dt} = a\sin t$

$$A = 2\pi\int_0^{2\pi} a(1-\cos t)\cdot \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\;dt$$

$$= 2\pi a\int_0^{2\pi}(1-\cos t)\sqrt{2a^2-2a^2\cos t}\,dt$$

$$= 2\sqrt{2}\pi a^2\int_0^{2\pi}(1-\cos t)^{3/2}\,dt$$

b. $1-\cos t = 2\sin^2\left(\frac{t}{2}\right)$, so

$$A = 2\sqrt{2}\pi a^2\int_0^{2\pi} 2^{3/2}\sin^3\left(\frac{t}{2}\right)dt$$

$$= 8\pi a^2\int_0^{2\pi}\sin\left(\frac{t}{2}\right)\sin^2\left(\frac{t}{2}\right)dt$$

$$= 8\pi a^2\int_0^{2\pi}\sin\left(\frac{t}{2}\right)\left[1-\cos^2\left(\frac{t}{2}\right)\right]dt$$

$$= 8\pi a^2\left[-2\cos\left(\frac{t}{2}\right)+\frac{2}{3}\cos^3\left(\frac{t}{2}\right)\right]_0^{2\pi}$$

$$= 8\pi a^2\left[\left(2-\frac{2}{3}\right)-\left(-2+\frac{2}{3}\right)\right] = \frac{64}{3}\pi a^2$$

34. $\frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = a\cos t$

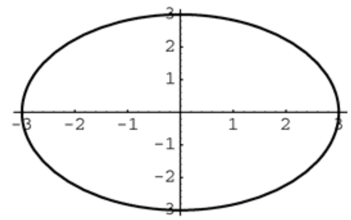
Since the circle is being revolved about the line $x = b$, the surface area is

$$A = 2\pi\int_0^{2\pi}(b-a\cos t)\sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt$$

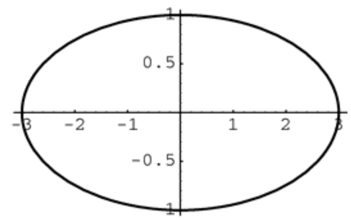
$$= 2\pi a\int_0^{2\pi}(b-a\cos t)\,dt$$

$$= 2\pi a[bt - a\sin t]_0^{2\pi} = 4\pi^2 ab$$

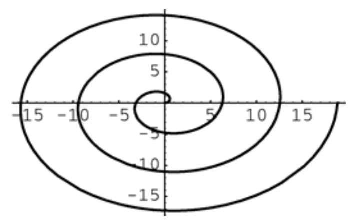
35. a.



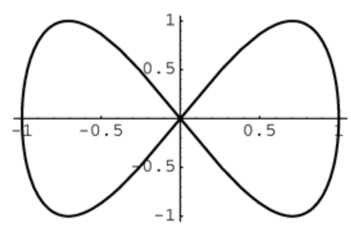
b.



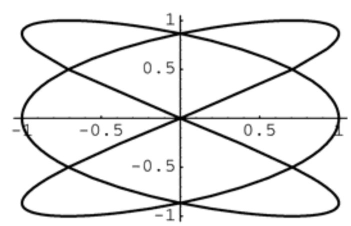
c.



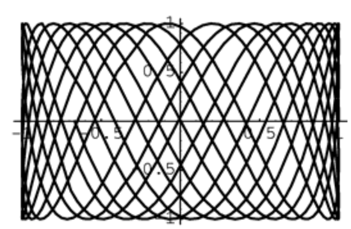
d.



e.



f.



36. **a.** $f'(t) = -3\sin t,\ g'(t) = 3\cos t$

$L = \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$

$= \int_0^{2\pi} 3dt = 3[t]_0^{2\pi} = 6\pi \approx 18.850$

b. $f'(t) = -3\sin t,\ g'(t) = \cos t$

$L = \int_0^{2\pi} \sqrt{9\sin^2 t + \cos^2 t}\, dt \approx 13.365$

c. $f'(t) = \cos t - t\sin t,\ g'(t) = t\cos t + \sin t$

$L = \int_0^{6\pi} \sqrt{(\cos t - t\sin t)^2 + (t\cos t + \sin t)^2}\, dt$

$= \int_0^{6\pi} \sqrt{1+t^2}\, dt \approx 179.718$

d. $f'(t) = -\sin t,\ g'(t) = 2\cos 2t$

$L = \int_0^{2\pi} \sqrt{\sin^2 t + 4\cos^2 2t}\, dt \approx 9.429$

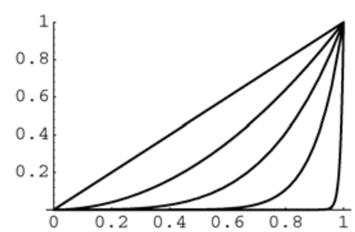
e. $f'(t) = -3\sin 3t,\ g'(t) = 2\cos 2t$

$L = \int_0^{2\pi} \sqrt{9\sin^2 3t + 4\cos^2 2t}\, dt \approx 15.289$

f. $f'(t) = -\sin t,\ g'(t) = \pi\cos \pi t$

$L = \int_0^{40} \sqrt{\sin^2 t + \pi^2\cos^2 \pi t}\, dt \approx 86.58$

37.



$y = x,\ y' = 1,$

$L = \int_0^1 \sqrt{2}dx = \left[\sqrt{2}\, x\right]_0^1 = \sqrt{2} \approx 1.41421$

$y = x^2,\ y' = 2x,\ L = \int_0^1 \sqrt{1+4x^2}\, dx \approx 1.47894$

$y = x^4,\ y' = 4x^3,\ L = \int_0^1 \sqrt{1+16x^6}\, dx \approx 1.60023$

$y = x^{10},\ y' = 10x^9,$

$L = \int_0^1 \sqrt{1+100^{18}}\, dx \approx 1.75441$

$y = x^{100},\ y' = 100x^{99},$

$L = \int_0^1 \sqrt{1+10{,}000x^{198}}\, dx \approx 1.95167$

When n = 10,000 the length will be close to 2.

5.5 Concepts Review

1. $F \cdot (b-a); \int_a^b F(x)\, dx$

2. $30 \cdot 10 = 300$

3. the depth of that part of the surface

4. δhA

Problem Set 5.5

1. $F\left(\frac{1}{2}\right) = 6;\ k \cdot \frac{1}{2} = 6,\ k = 12$

$F(x) = 12x$

$W = \int_0^{1/2} 12x\, dx = \left[6x^2\right]_0^{1/2} = \frac{3}{2} = 1.5$ ft-lb

2. From Problem 1, $F(x) = 12x$.

$W = \int_0^2 12x\, dx = \left[6x^2\right]_0^2 = 24$ ft-lb

3. $F(0.01) = 0.6;\ k = 60$

$F(x) = 60x$

$W = \int_0^{0.02} 60x\, dx = \left[30x^2\right]_0^{0.02} = 0.012$ Joules

4. $F(x) = kx$ and let l be the natural length of the spring.

$W = \int_{8-l}^{9-l} kx\, dx = \left[\frac{1}{2}kx^2\right]_{8-l}^{9-l}$

$= \frac{1}{2}k\left[(81 - 18l + l^2) - (64 - 16l + l^2)\right]$

$= \frac{1}{2}k(17 - 2l) = 0.05$

Thus, $k = \frac{0.1}{17 - 2l}$.

$W = \int_{9-l}^{10-l} kx\, dx = \left[\frac{1}{2}kx^2\right]_{9-l}^{10-l}$

$= \frac{1}{2}k\left[(100 - 20l + l^2) - (81 - 18l + l^2)\right]$

$= \frac{1}{2}k(19 - 2l) = 0.1$

Thus, $k = \frac{0.2}{19 - 2l}$.

Solving $\frac{0.1}{17 - 2l} = \frac{0.2}{19 - 2l},\ l = \frac{15}{2}$.

Thus $k = 0.05$, and the natural length is 7.5 cm.

5. $W = \int_0^d kx\,dx = \left[\frac{1}{2}kx^2\right]_0^d$

$= \frac{1}{2}k(d^2 - 0) = \frac{1}{2}kd^2$

6. $F(8) = 2; k16 = 2, k = \frac{1}{8}$

$W = \int_0^{27} \frac{1}{8}s^{4/3}\,ds = \frac{1}{8}\left[\frac{3}{7}s^{7/3}\right]_0^{27} = \frac{6561}{56}$

$\approx$ 117.16 inch-pounds

7. $W = \int_0^2 9s\,ds = 9\left[\frac{1}{2}s^2\right]_0^2 = 18$ ft-lb

8. One spring will move from 2 feet beyond its natural length to 3 feet beyond its natural length. The other will move from 2 feet beyond its natural length to 1 foot beyond its natural length.

$W = \int_2^3 6s\,ds + \int_2^1 6s\,ds = \left[3s^2\right]_2^3 + \left[3s^2\right]_2^1$

= 3(9 – 4) + 3(1 – 4) = 6 ft-lb

9. A slab of thickness Δy at height y has width $4 - \frac{4}{5}y$ and length 10. The slab will be lifted a distance $10 - y$.

$\Delta W \approx \delta \cdot 10 \cdot \left(4 - \frac{4}{5}y\right)\Delta y(10 - y)$

$= 8\delta(y^2 - 15y + 50)\Delta y$

$W = \int_0^5 8\delta(y^2 - 15y + 50)\,dy$

$= 8(62.4)\left[\frac{1}{3}y^3 - \frac{15}{2}y^2 + 50y\right]_0^5$

$= 8(62.4)\left(\frac{125}{3} - \frac{375}{2} + 250\right) = 52{,}000$ ft-lb

10. A slab of thickness Δy at height y has width $4 - \frac{4}{3}y$ and length 10. The slab will be lifted a distance $8 - y$.

$\Delta W \approx \delta \cdot 10 \cdot \left(4 - \frac{4}{3}y\right)\Delta y(8 - y)$

$= \frac{40}{3}\delta(24 - 11y + y^2)\Delta y$

$W = \int_0^3 \frac{40}{3}\delta(24 - 11y + y^2)\,dy$

$= \frac{40}{3}(62.4)\left[24y - \frac{11}{2}y^2 + \frac{1}{3}y^3\right]_0^3$

$= \frac{40}{3}(62.4)\left(72 - \frac{99}{2} + 9\right) = 26{,}208$ ft-lb

11. A slab of thickness Δy at height y has width $\frac{3}{4}y + 3$ and length 10. The slab will be lifted a distance $9 - y$. $\Delta W \approx \delta \cdot 10 \cdot \left(\frac{3}{4}y + 3\right)\Delta y(9 - y)$

$= \frac{15}{2}\delta(36 + 5y - y^2)\Delta y$

$W = \int_0^4 \frac{15}{2}\delta(36 + 5y - y^2)\,dy$

$= \frac{15}{2}(62.4)\left[36y + \frac{5}{2}y^2 - \frac{1}{3}y^3\right]_0^4$

$= \frac{15}{2}(62.4)\left(144 + 40 - \frac{64}{3}\right)$

= 76,128 ft-lb

12. A slab of thickness Δy at height y has width $2\sqrt{6y - y^2}$ and length 10. The slab will be lifted a distance $8 - y$.

$\Delta W \approx \delta \cdot 10 \cdot 2\sqrt{6y - y^2}\,\Delta y(8 - y)$

$= 20\delta\sqrt{6y - y^2}\,(8 - y)\Delta y$

$W = \int_0^3 20\delta\sqrt{6y - y^2}\,(8 - y)\,dy$

$= 20\delta\int_0^3 \sqrt{6y - y^2}\,(3 - y)\,dy$

$+ 20\delta\int_0^3 \sqrt{6y - y^2}\,(5)\,dy$

$= 20\delta\left[\frac{1}{3}(6y - y^2)^{3/2}\right]_0^3 + 100\delta\int_0^3 \sqrt{6y - y^2}\,dy$

Notice that $\int_0^3 \sqrt{6y - y^2}\,dy$ is the area of a quarter of a circle with radius 3.

$W = 20\delta(9) + 100\delta\left(\frac{1}{4}\pi 9\right)$

$= (62.4)(180 + 225\pi) \approx 55{,}340$ ft-lb

13. The volume of a disk with thickness Δy is $16\pi\Delta y$. If it is at height y, it will be lifted a distance $10 - y$.

$\Delta W \approx \delta 16\pi\Delta y(10 - y) = 16\pi\delta(10 - y)\Delta y$

$W = \int_0^{10} 16\pi\delta(10 - y)\,dy = 16\pi(50)\left[10y - \frac{1}{2}y^2\right]_0^{10}$

$= 16\pi(50)(100 - 50) \approx 125{,}664$ ft-lb

14. The volume of a disk with thickness Δx at height x is $\pi(4+x)^2\Delta x$. It will be lifted a distance of $10 - x$.

$$\Delta W \approx \delta\pi(4+x)^2 \Delta x(10-x)$$
$$= \pi\delta(160+64x+2x^2-x^3)\Delta x$$
$$W = \int_0^{10} \pi\delta(160+64x+2x^2-x^3)dx$$
$$= \pi(50)\left[160x+32x^2+\frac{2}{3}x^3-\frac{1}{4}x^4\right]_0^{10}$$
$$= \pi(50)\left(1600+3200+\frac{2000}{3}-2500\right)$$
$$\approx 466{,}003 \text{ ft-lb}$$

15. The total force on the face of the piston is $A \cdot f(x)$ if the piston is x inches from the cylinder head. The work done by moving the piston from x_1 to x_2 is $W = \int_{x_1}^{x_2} A \cdot f(x)dx = A\int_{x_1}^{x_2} f(x)dx$.
This is the work done by the gas in moving the piston. The work done by the piston to compress the gas is the opposite of this or $A\int_{x_2}^{x_1} f(x)dx$.

16. $c = 40(16)^{1.4}$

$A = 1;\ p(v) = cv^{-1.4}$

$f(x) = cx^{-1.4}$

$x_1 = \frac{16}{1} = 16,\ x_2 = \frac{2}{1} = 2$

$$W = \int_2^{16} cx^{-1.4}dx = c\left[-2.5x^{-0.4}\right]_2^{16}$$
$$= 40(16)^{1.4}(-2.5)(16^{-0.4}-2^{-0.4})$$
$$\approx 2075.83 \text{ in.-lb}$$

17. $c = 40(16)^{1.4}$

$A = 2;\ p(v) = cv^{-1.4}$

$f(x) = c(2x)^{-1.4}$

$x_1 = \frac{16}{2} = 8,\ x_2 = \frac{2}{2} = 1$

$$W = 2\int_1^{8} c(2x)^{-1.4}dx = 2c\left[-1.25(2x)^{-0.4}\right]_1^{8}$$
$$= 80(16)^{1.4}(-1.25)(16^{-0.4}-2^{-0.4})$$
$$\approx 2075.83 \text{ in.-lb}$$

18. 80 lb/in.^{2} = 11,520 lb/ft^2

$c = 11{,}520(1)^{1.4} = 11{,}520$

$$\Delta W \approx p(v)\Delta v = 11{,}520v^{-1.4}\Delta v$$
$$W = \int_1^4 11{,}520v^{-1.4}dv = \left[-28{,}800v^{-0.4}\right]_1^4$$
$$= -28{,}800(4^{-0.4}-1^{-0.4}) \approx 12{,}259 \text{ ft-lb}$$

19. The total work is equal to the work W_1 to haul the load by itself and the work W_2 to haul the rope by itself.

$W_1 = 200 \cdot 500 = 100{,}000$ ft-lb

Let $y = 0$ be the bottom of the shaft. When the rope is at y, $\Delta W_2 \approx 2\Delta y(500-y)$.

$$W_2 = \int_0^{500} 2(500-y)dy = 2\left[500y-\frac{1}{2}y^2\right]_0^{500}$$
$$= 2(250{,}000 - 125{,}000) = 250{,}000 \text{ ft-lb}$$
$$W = W_1 + W_2 = 100{,}000 + 250{,}000$$
$$= 350{,}000 \text{ ft-lb}$$

20. The total work is equal to the work W_1 to lift the monkey plus the work W_2 to lift the chain.

$W_1 = 10 \cdot 20 = 200$ ft-lb

Let $y = 20$ represent the top. As the monkey climbs the chain, the piece of chain at height y $(0 \le y \le 10)$ will be lifted $20 - 2y$ ft.

$$\Delta W_2 \approx \frac{1}{2}\Delta y(20-2y) = (10-y)\Delta y$$
$$W_2 = \int_0^{10}(10-y)dy = \left[10y-\frac{1}{2}y^2\right]_0^{10}$$
$$= 100 - 50 = 50 \text{ ft-lb}$$
$$W = W_1 + W_2 = 250 \text{ ft-lb}$$

21. $f(x) = \frac{k}{x^2};\ f(4000) = 5000$

$$\frac{k}{4000^2} = 5000,\ k = 80{,}000{,}000{,}000$$
$$W = \int_{4000}^{4200} \frac{80{,}000{,}000{,}000}{x^2}dx$$
$$= 80{,}000{,}000{,}000\left[-\frac{1}{x}\right]_{4000}^{4200}$$
$$= \frac{20{,}000{,}000}{21} \approx 952{,}381 \text{ mi-lb}$$

22. $F(x) = \frac{k}{x^2}$ where x is the distance between the charges. $F(2) = 10;\ \frac{k}{4} = 10,\ k = 40$

$$W = \int_1^5 \frac{40}{x^2}dx = \left[-\frac{40}{x}\right]_1^5 = 32 \text{ ergs}$$

23. The relationship between the height of the bucket and time is $y = 2t$, so $t = \frac{1}{2}y$. When the bucket is a height y, the sand has been leaking out of the bucket for $\frac{1}{2}y$ seconds. The weight of the bucket and sand is $100 + 500 - 3\left(\frac{1}{2}y\right) = 600 - \frac{3}{2}y.$

$$\Delta W \approx \left(600 - \frac{3}{2}y\right)\Delta y$$

$$W = \int_0^{80}\left(600 - \frac{3}{2}y\right)dy = \left[600y - \frac{3}{4}y^2\right]_0^{80}$$

$= 48{,}000 – 4800 = 43{,}200$ ft-lb

24. The total work is equal to the work W_1 needed to fill the pipe plus the work W_2 needed to fill the tank.

$$\Delta W_1 = \delta\pi\left(\frac{1}{2}\right)^2 \Delta y(y) = \frac{\delta\pi y}{4}\Delta y$$

$$W_1 = \int_0^{30}\frac{\delta\pi y}{4}dy = \frac{(62.4)\pi}{4}\left[\frac{1}{2}y^2\right]_0^{30}$$

$\approx 22{,}054$ ft-lb

The cross sectional area at height y feet $(30 \le y \le 50)$ is πr^2 where

$r = \sqrt{10^2 - (40-y)^2} = \sqrt{-y^2 + 80y - 1500}$.

$$\Delta W_2 = \delta\pi r^2 \Delta y\, y = \delta\pi(-y^3 + 80y^2 - 1500y)\Delta y$$

$$W_2 = \int_{30}^{50}\delta\pi(-y^3 + 80y^2 - 1500y)dy$$

$$= (62.4)\pi\left[-\frac{1}{4}y^4 + \frac{80}{3}y^3 - 750y^2\right]_{30}^{50}$$

$$= (62.4)\pi\left[\left(-1{,}562{,}500 + \frac{10{,}000{,}000}{3} - 1{,}875{,}000\right) - (-202{,}500 + 720{,}000 - 675{,}000)\right]$$

$\approx$ 10,455,220 ft-lb

$W = W_1 + W_2 \approx 10{,}477{,}274$ ft-lb

25. Let y measure the height of a narrow rectangle with $0 \le y \le 3$. The force against this rectangle at depth $3 – y$ is $\Delta F \approx \delta(3-y)(6)\Delta y$. Thus,

$$F = \int_0^3 \delta(3-y)(6)\,dy = 6\delta\left[3y - \frac{y^2}{2}\right]_0^3$$

$= 6 \cdot 62.4(4.5) = 1684.8$ pounds

26. Let y measure the height of a narrow rectangle with $0 \le y \le 3$. The force against this rectangle at depth $5 – y$ is $\Delta F \approx \delta(5-y)(6)\Delta y$. Thus,

$$F = \int_0^3 \delta(5-y)(6)\,dy = 6\delta\left[5y - \frac{y^2}{2}\right]_0^3$$

$= 6 \cdot 62.4 \cdot 10.5 = 3931.2$ pounds

27. Place the equilateral triangle in the coordinate system such that the vertices are $(-3,0), (3,0)$ and $\left(0, -3\sqrt{3}\right)$.

The equation of the line in Quadrant I is

$y = \sqrt{3}\cdot x - 3\sqrt{3}$ or $x = \frac{y}{\sqrt{3}} + 3.$

$\Delta F \approx \delta(-y)\left(2\left(\frac{y}{\sqrt{3}} + 3\right)\right)\Delta y$ and

$$F = \int_{-3\sqrt{3}}^0 \delta(-y)\left(2\left(\frac{y}{\sqrt{3}} + 3\right)\right)dy$$

$$= -2\delta\int_{-3\sqrt{3}}^0\left(\frac{y^2}{\sqrt{3}} + 3y\right)dy$$

$$= -2\delta\left[\frac{y^3}{3\sqrt{3}} + \frac{3y^2}{2}\right]_{-3\sqrt{3}}^0 = -2\cdot 62.4(0 - 13.5)$$

$= 1684.8$ pounds

28. Place the right triangle in the coordinate system such that the vertices are (0,0), (3,0) and (0,-4). The equation of the line in Quadrant IV is

$y = \frac{4}{3}x - 4$ or $x = \frac{3}{4}y + 3.$

$\Delta F \approx \delta(3-y)\left(\frac{3}{4}y + 3\right)\Delta y$ and

$$F = \int_{-4}^0 \delta\left(9 - \frac{3}{4}y - \frac{3}{4}y^2\right)dy$$

$$= \delta\left[9y - \frac{3y^2}{8} - \frac{y^3}{4}\right]_{-4}^0 = 62.4 \cdot 26$$

$= 1622.4$ pounds

29. $\Delta F \approx \delta(1-y)\left(\sqrt{y}\right)\Delta y;\ F = \int_0^1 \delta(1-y)\left(\sqrt{y}\right)dy$

$$= \delta\int_0^1\left(y^{1/2} - y^{3/2}\right)dy$$

$$= \delta\left[\frac{2}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1 = 62.4\left(\frac{4}{15}\right)$$

$= 16.64$ pounds

30. Place the circle in the coordinate system so that the center is (0.0). The equation of the circle is $x^2 + y^2 = 16$ and in Quadrants I and IV,

$x = \sqrt{16 - y^2}$. $\Delta F \approx \delta(6-y)\left(2\sqrt{16-y^2}\right)\Delta y$

$$F = \int_{-4}^{4} \delta(6-y)\left(2\sqrt{16-y^2}\right)dy$$

Using a CAS, $F \approx 18{,}819$ pounds.

31. Place a rectangle in the coordinate system such that the vertices are (0,0), (0,b), (a,0) and (a,b). The equation of the diagonal from (0,0) to (a,b) is $y = \frac{b}{a}x$ or $x = \frac{a}{b}y$. For the upper left triangle I,

$\Delta F \approx \delta(b-y)\left(\frac{a}{b}y\right)\Delta y$ and

$$F = \int_0^b \delta(b-y)\left(\frac{a}{b}y\right)dy$$

$$= \delta\int_0^b \left(y - \frac{a}{b}y^2\right)dy = \delta\left[\frac{ay^2}{2} - \frac{ay^3}{3b}\right]_0^b$$

$$= \delta\left(\frac{ab^2}{2} - \frac{ab^2}{3}\right) = \delta\frac{ab^2}{6}$$

For the lower right triangle II,

$\Delta F \approx \delta(b-y)\left(a - \frac{a}{b}y\right)dy$ and

$$F = \int_0^b \delta(b-y)\left(a - \frac{a}{b}y\right)dy$$

$$= \int_0^b \delta\left(ab - 2ay + \frac{a}{b}y^2\right)dy$$

$$= \delta\left[aby - ay^2 + \frac{ay^3}{3b}\right]_0^b = \delta\left(ab^2 - ab^2 + \frac{ab^2}{3}\right)$$

$$= \delta\frac{ab^2}{3}$$

The total force on one half of the dam is twice the total force on the other half since $\dfrac{\delta\frac{ab^2}{3}}{\delta\frac{ab^2}{6}} = 2.$

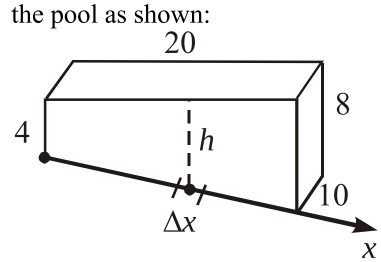
32. Consider one side of the cube and place the vertices of this square on (0,0), (0,2), (2,0) and (2,2).

$\Delta F \approx \delta(102-y)(2)\Delta y$; $F = \int_0^2 2\delta(102-y)\,dy$

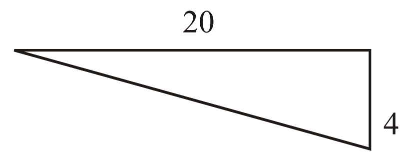
$$= 2\delta\left[102y - \frac{y^2}{2}\right]_0^2 = 2\cdot 62.4\cdot 202 = 25{,}209.6$$

The force on all six sides would be 6(25,209.6) = 151,257.6 pounds.

33. We can position the x-axis along the bottom of the pool as shown:



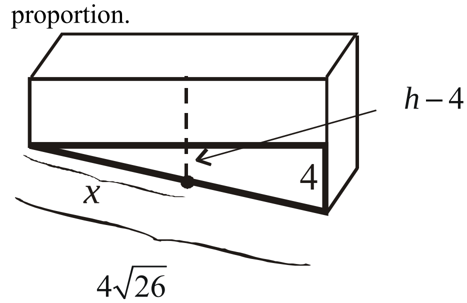
From the diagram, we let h = the depth of an arbitrary slice along the width of the bottom of the pool.



Using the Pythagorean Theorem, we can find that the length of the bottom of the pool is

$\sqrt{20^2 + 4^2} = \sqrt{416} = 4\sqrt{26}$

Next, we need to get h in terms of x. This can be done by using similar triangles to set up a proportion.



$$\frac{h-4}{4} = \frac{x}{4\sqrt{26}} \quad \rightarrow \quad h = 4 + \frac{x}{\sqrt{26}}$$

$$\Delta F = \delta\cdot h\cdot\Delta A$$

$$F = \int_0^{4\sqrt{26}} \delta\left(4 + \frac{x}{\sqrt{26}}\right)(10)\,dx$$

$$= \int_0^{4\sqrt{26}} 62.4\left(4 + \frac{x}{\sqrt{26}}\right)(10)\,dx$$

$$= 624\int_0^{4\sqrt{26}} \left(4 + \frac{x}{\sqrt{26}}\right)dx$$

$$= 624\left[4x + \frac{x^2}{2\sqrt{26}}\right]_0^{4\sqrt{26}}$$

$$= 624\left(16\sqrt{26} + 8\sqrt{26}\right) = 624\left(24\sqrt{26}\right)$$

$$= 14{,}976\sqrt{26}\ \text{lb}\quad (\approx 76{,}362.92\ \text{lb})$$

34. If we imagine unrolling the cylinder so we have a flat sheet, then we need to find the total force against one side of a rectangular plate as if it had been submerged in the oil. The rectangle would be $2\pi(5) = 10\pi$ feet wide and 6 feet high.

Thus, the total lateral force is given by

$$F = \int_0^6 50 \cdot y \cdot 10\pi\, dy$$

$$= 500\pi \int_0^6 y\, dy = \left[250\pi y^2\right]_0^6$$

$$= 250\pi(36) = 9000\pi \text{ lbs} \quad (\approx 28,274.33 \text{ lb})$$

35. Let W_1 be the work to lift V to the surface and W_2 be the work to lift V from the surface to 15 feet above the surface. The volume displaced by the buoy y feet above its original position is

$$\frac{1}{3}\pi\left(a - \frac{a}{h}y\right)^2 (h-y) = \frac{1}{3}\pi a^2 h\left(1 - \frac{y}{h}\right)^3.$$

The weight displaced is $\frac{\delta}{3}\pi a^2 h\left(1 - \frac{y}{h}\right)^3$.

Note by Archimede's Principle $m = \frac{\delta}{3}\pi a^2 h$ or

$a^2 h = \frac{3m}{\delta\pi}$, so the displaced weight is

$$m\left(1 - \frac{y}{h}\right)^3.$$

$$\Delta W_1 \approx \left(m - m\left(1 - \frac{y}{h}\right)^3\right)\Delta y = m\left(1 - \left(1 - \frac{y}{h}\right)^3\right)\Delta y$$

$$W_1 = m\int_0^h \left(1 - \left(1 - \frac{y}{h}\right)^3\right) dy$$

$$= m\left[y + \frac{h}{4}\left(1 - \frac{y}{h}\right)^4\right]_0^h = \frac{3mh}{4}$$

$$W_2 = m \cdot 15 = 15m$$

$$W = W_1 + W_2 = \frac{3mh}{4} + 15m$$

36. First calculate the work W_1 needed to lift the contents of the bottom tank to 10 feet.

$$\Delta W_1 \approx \delta 40 \Delta y (10 - y)$$

$$W_1 = \int_0^4 \delta 40(10 - y)\, dy$$

$$= (62.4)(40)\left[-\frac{1}{2}(10 - y)^2\right]_0^4$$

$= (62.4)(40)(–18 + 50) = 79,872$ ft-lb

Next calculate the work W_2 needed to fill the top tank. Let y be the distance from the bottom of the top tank.

$$\Delta W_2 \approx \delta(36\pi)\Delta y\, y$$

Solve for the height of the top tank:

$$36\pi h = 160;\ h = \frac{160}{36\pi} = \frac{40}{9\pi}$$

$$W_2 = \int_0^{40/9\pi} \delta 36\pi y\, dy$$

$$= (62.4)(36\pi)\left[\frac{1}{2}y^2\right]_0^{40/9\pi}$$

$$= (62.4)(36\pi)\left(\frac{800}{81\pi^2}\right) \approx 7062 \text{ ft-lbs}$$

$$W = W_1 + W_2 \approx 86,934 \text{ ft-lbs}$$

37. Since $\delta\left(\frac{1}{3}\pi a^2\right)(8) = 300,\ a = \sqrt{\frac{225}{2\pi\delta}}$.

When the buoy is at z feet ($0 \le z \le 2$) below floating position, the radius r at the water level is

$$r = \left(\frac{8+z}{8}\right)a = \sqrt{\frac{225}{2\pi\delta}}\left(\frac{8+z}{8}\right).$$

$$F = \delta\left(\frac{1}{3}\pi r^2\right)(8+z) - 300$$

$$= \frac{75}{128}(8+z)^3 - 300$$

$$W = \int_0^2 \left[\frac{75}{128}(8+z)^3 - 300\right] dz$$

$$= \left[\frac{75}{512}(8+z)^4 - 300z\right]_0^2$$

$$= \left(\frac{46,875}{32} - 600\right) - (600 - 0)$$

$$= \frac{8475}{32} \approx 264.84 \text{ ft-lb}$$

5.6 Concepts Review

1. right; $\frac{4 \cdot 1+6 \cdot 3}{4+6}=2.2$

2. 2.5; right; $x(1+x)$; $1+x$

3. 1; 3

4. $\frac{24}{16}; \frac{40}{16}$
The second lamina balances at $\bar{x}=3,\ \bar{y}=1$.
The first lamina has area 12 and the second lamina has area 4.
$\bar{x}=\frac{12 \cdot 1+4 \cdot 3}{12+4}=\frac{24}{16}, \bar{y}=\frac{12 \cdot 3+4 \cdot 1}{12+4}=\frac{40}{16}$

Problem Set 5.6

1. $\bar{x}=\frac{2 \cdot 5+(-2) \cdot 7+1 \cdot 9}{5+7+9}=\frac{5}{21}$

2. Let x measure the distance from the end where John sits.
$\frac{180 \cdot 0+80 \cdot x+110 \cdot 12}{180+80+110}=6$
$80x + 1320 = 6 \cdot 370$
$80x = 900$
$x = 11.25$
Tom should be 11.25 feet from John, or, equivalently, 0.75 feet from Mary.

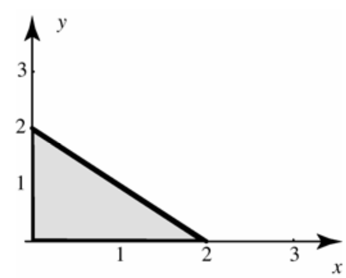
3. $\bar{x}=\frac{\int_0^7 x\sqrt{x}\,dx}{\int_0^7 \sqrt{x}\,dx}=\frac{\left[\frac{2}{5}x^{5/2}\right]_0^7}{\left[\frac{2}{3}x^{3/2}\right]_0^7}=\frac{\frac{2}{5}\left(49\sqrt{7}\right)}{\frac{2}{3}\left(7\sqrt{7}\right)}=\frac{21}{5}$

4. $\bar{x}=\frac{\int_0^7 x(1+x^3)dx}{\int_0^7 (1+x^3)dx}=\frac{\left[\frac{1}{2}x^2+\frac{1}{5}x^5\right]_0^7}{\left[x+\frac{1}{4}x^4\right]_0^7}$
$=\frac{\left(\frac{49}{2}+\frac{16{,}807}{5}\right)}{\left(7+\frac{2401}{4}\right)}=\frac{\frac{33{,}859}{10}}{\frac{2429}{4}}=\frac{9674}{1735}\approx 5.58$

5. $M_y=1 \cdot 2+7 \cdot 3+(-2) \cdot 4+(-1) \cdot 6+4 \cdot 2=17$
$M_x=1 \cdot 2+1 \cdot 3+(-5) \cdot 4+0 \cdot 6+6 \cdot 2=-3$
$m = 2 + 3 + 4 + 6 + 2 = 17$
$\bar{x}=\frac{M_y}{m}=1, \bar{y}=\frac{M_x}{m}=-\frac{3}{17}$

6. $M_y=(-3) \cdot 5+(-2) \cdot 6+3 \cdot 2+4 \cdot 7+7 \cdot 1=14$
$M_x=2 \cdot 5+(-2) \cdot 6+5 \cdot 2+3 \cdot 7+(-1) \cdot 1=28$
$m = 5 + 6 + 2 + 7 + 1 = 21$
$\bar{x}=\frac{M_y}{m}=\frac{2}{3}, \bar{y}=\frac{M_x}{m}=\frac{4}{3}$

7. Consider two regions R_1 and R_2 such that R_1 is bounded by *f(x)* and the *x*-axis, and R_2 is bounded by *g(x)* and the *x*-axis. Let R_3 be the region formed by $R_1 - R_2$. Make a regular partition of the homogeneous region R_3 such that each sub-region is of width , Δx and let x be the distance from the *y*-axis to the center of mass of a sub-region. The heights of R_1 and R_2 at x are approximately *f(x)* and *g(x)* respectively. The mass of R_3 is approximately
$$\begin{aligned}\Delta m &= \Delta m_1 - \Delta m_2 \\ &\approx \delta f(x)\Delta x - \delta g(x)\Delta x \\ &= \delta[f(x)-g(x)]\Delta x\end{aligned}$$
where δ is the density. The moments for R_3 are approximately
$$\begin{aligned}M_x &= M_x(R_1)-M_x(R_2) \\ &\approx \frac{\delta}{2}[f(x)]^2\Delta x-\frac{\delta}{2}[g(x)]^2\Delta x \\ &= \frac{\delta}{2}\left[(f(x))^2-(g(x))^2\right]\Delta x\end{aligned}$$
$$\begin{aligned}M_y &= M_y(R_1)-M_y(R_2) \\ &\approx x\delta f(x)\Delta x - x\delta g(x)\Delta x \\ &= x\delta[f(x)-g(x)]\Delta x\end{aligned}$$
Taking the limit of the regular partition as $\Delta x \to 0$ yields the resulting integrals in Figure 10.

8. 

$f(x)=2-x; g(x)=0$
$$\bar{x}=\frac{\int_0^2 x[(2-x)-0]dx}{\int_0^2 [(2-x)-0]dx}$$
$$=\frac{\int_0^2 [2x-x^2]dx}{\int_0^2 [2-x]dx}$$

$$= \frac{\left(x^2 - \frac{1}{3}x^3\right)_0^2}{\left(2x - \frac{1}{2}x^2\right)_0^2} = \frac{4 - \frac{8}{3}}{4 - 2}$$

$$= \frac{2}{3}$$

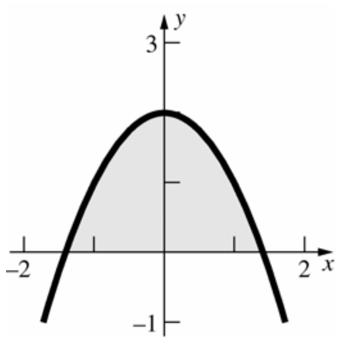
$$\overline{y} = \frac{\frac{1}{2}\int_0^2 [(2-x)^2 - 0^2]dx}{\int_0^2 [(2-x) - 0]dx}$$

$$= \frac{\int_0^2 [4 - 4x + x^2]dx}{4}$$

$$= \frac{\left(4x - 2x^2 + \frac{1}{3}x^3\right)_0^2}{4} = \frac{8 - 8 + \frac{8}{3}}{4}$$

$$= \frac{2}{3}$$

9.



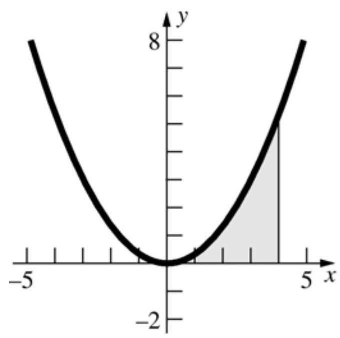
$\overline{x} = 0$ (by symmetry)

$$\overline{y} = \frac{\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2)^2\, dx}{\int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2)dx}$$

$$= \frac{\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}} (4 - 4x^2 + x^4)dx}{\left[2x - \frac{1}{3}x^3\right]_{-\sqrt{2}}^{\sqrt{2}}}$$

$$= \frac{\frac{1}{2}\left[4x - \frac{4}{3}x^3 + \frac{1}{5}x^5\right]_{-\sqrt{2}}^{\sqrt{2}}}{\frac{8\sqrt{2}}{3}} = \frac{\frac{32\sqrt{2}}{15}}{\frac{8\sqrt{2}}{3}} = \frac{4}{5}$$

10.



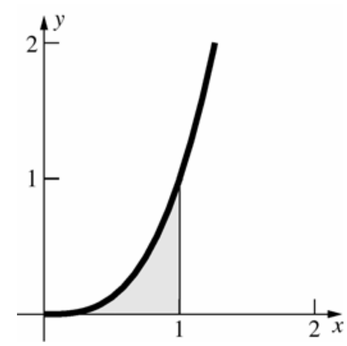
$$\overline{x} = \frac{\int_0^4 x\left(\frac{1}{3}x^2\right)dx}{\int_0^4 \frac{1}{3}x^2 dx} = \frac{\frac{1}{3}\int_0^4 x^3 dx}{\frac{1}{3}\int_0^4 x^2\, dx}$$

$$= \frac{\frac{1}{3}\left[\frac{1}{4}x^4\right]_0^4}{\frac{1}{3}\left[\frac{1}{3}x^3\right]_0^4} = \frac{\frac{64}{3}}{\frac{64}{9}} = 3$$

$$\overline{y} = \frac{\frac{1}{2}\int_0^4 \left(\frac{1}{3}x^2\right)^2 dx}{\int_0^4 \frac{1}{3}x^2\, dx} = \frac{\frac{1}{18}\int_0^4 x^4 dx}{\frac{64}{9}} = \frac{\frac{1}{18}\left[\frac{1}{5}x^5\right]_0^4}{\frac{64}{9}}$$

$$= \frac{\frac{512}{45}}{\frac{64}{9}} = \frac{8}{5}$$

11.

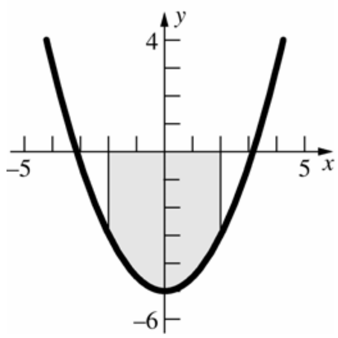


$$\overline{x} = \frac{\int_0^1 x(x^3)dx}{\int_0^1 x^3 dx} = \frac{\int_0^1 x^4\, dx}{\left[\frac{1}{4}x^4\right]_0^1} = \frac{\left[\frac{1}{5}x^5\right]_0^1}{\frac{1}{4}} = \frac{\frac{1}{5}}{\frac{1}{4}} = \frac{4}{5}$$

$$\overline{y} = \frac{\frac{1}{2}\int_0^1 (x^3)^2 dx}{\int_0^1 x^3\, dx} = \frac{\frac{1}{2}\int_0^1 x^6 dx}{\frac{1}{4}} = \frac{\left[\frac{1}{14}x^7\right]_0^1}{\frac{1}{4}}$$

$$= \frac{\frac{1}{14}}{\frac{1}{4}} = \frac{2}{7}$$

12.



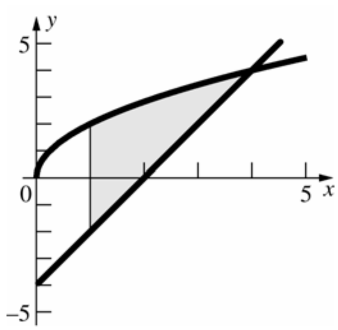
$\overline{x} = 0$ (by symmetry)

$$\overline{y} = \frac{\frac{1}{2}\int_{-2}^{2}\left[-\left(\frac{1}{2}(x^2-10)\right)^2\right]dx}{\int_{-2}^{2}\left[-\frac{1}{2}(x^2-10)\right]dx}$$

$$= \frac{-\frac{1}{8}\int_{-2}^{2}(x^4-20x^2+100)dx}{-\frac{1}{2}\int_{-2}^{2}(x^2-10)dx}$$

$$= \frac{-\frac{1}{8}\left[\frac{1}{5}x^5-\frac{20}{3}x^3+100x\right]_{-2}^{2}}{-\frac{1}{2}\left[\frac{1}{3}x^3-10x\right]_{-2}^{2}} = \frac{-\frac{574}{15}}{\frac{52}{3}} = -\frac{287}{130}$$

13.



To find the intersection point, solve

$2x-4 = 2\sqrt{x}$.

$x-2 = \sqrt{x}$

$x^2-4x+4 = x$

$x^2-5x+4 = 0$

$(x-4)(x-1) = 0$

$x = 4$ ($x = 1$ is extraneous.)

$$\overline{x} = \frac{\int_1^4 x\left[2\sqrt{x}-(2x-4)\right]dx}{\int_1^4\left[2\sqrt{x}-(2x-4)\right]dx}$$

$$= \frac{2\int_1^4 (x^{3/2}-x^2+2x)dx}{2\int_1^4 (x^{1/2}-x+2)dx}$$

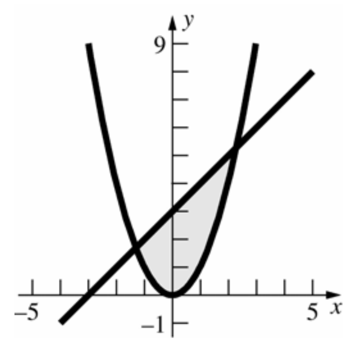
$$= \frac{2\left[\frac{2}{5}x^{5/2}-\frac{1}{3}x^3+x^2\right]_1^4}{2\left[\frac{2}{3}x^{3/2}-\frac{1}{2}x^2+2x\right]_1^4} = \frac{\frac{64}{5}}{\frac{19}{3}} = \frac{192}{95}$$

$$\overline{y} = \frac{\frac{1}{2}\int_1^4\left[\left(2\sqrt{x}\right)^2-(2x-4)^2\right]dx}{\int_1^4\left[2\sqrt{x}-\left(2x-4\right)\right]dx}$$

$$= \frac{2\int_1^4 (-x^2+5x-4)dx}{\frac{19}{3}}$$

$$= \frac{2\left[-\frac{1}{3}x^3+\frac{5}{2}x^2-4x\right]_1^4}{\frac{19}{3}} = \frac{9}{\frac{19}{3}} = \frac{27}{19}$$

14.



To find the intersection points, $x^2 = x+3$.

$x^2-x-3 = 0$

$x = \frac{1\pm\sqrt{13}}{2}$

$$\overline{x} = \frac{\int_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}} x(x+3-x^2)dx}{\int_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}} (x+3-x^2)dx}$$

$$= \frac{\int_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}} (x^2+3x-x^3)dx}{\left[\frac{1}{2}x^2+3x-\frac{1}{3}x^3\right]_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}}}$$

$$= \frac{\left[\frac{1}{3}x^3+\frac{3}{2}x^2-\frac{1}{4}x^4\right]_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}}}{\frac{13\sqrt{13}}{6}} = \frac{\frac{13\sqrt{3}}{12}}{\frac{13\sqrt{13}}{6}} = \frac{1}{2}$$

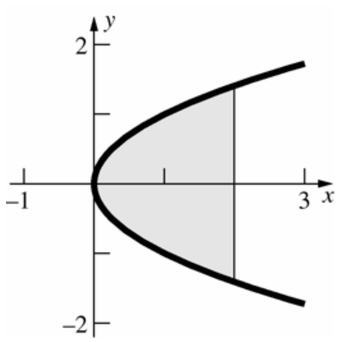
$$\overline{y} = \frac{\frac{1}{2}\int_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}}\left[(x+3)^2-(x^2)^2\right]dx}{\int_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}} (x+3-x^2)dx}$$

$$= \frac{\frac{1}{2}\int_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}} (x^2+6x+9-x^4)}{\frac{13\sqrt{13}}{6}}$$

$$= \frac{\frac{1}{2}\left[\frac{1}{3}x^3+3x^2+9x-\frac{1}{5}x^5\right]_{\frac{(1-\sqrt{13})}{2}}^{\frac{(1+\sqrt{13})}{2}}}{\frac{13\sqrt{13}}{6}}$$

$$= \frac{\frac{143\sqrt{13}}{30}}{\frac{13\sqrt{13}}{6}} = \frac{11}{5}$$

15.



To find the intersection points, solve $y^2 = 2$.

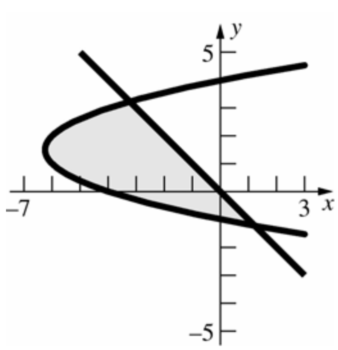
$y = \pm\sqrt{2}$

$$\overline{x} = \frac{\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}}\left[2^2-(y^2)^2\right]dy}{\int_{-\sqrt{2}}^{\sqrt{2}}(2-y^2)dy} = \frac{\frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}}(4-y^4)dy}{\left[2y-\frac{1}{3}y^3\right]_{-\sqrt{2}}^{\sqrt{2}}}$$

$$= \frac{\frac{1}{2}\left[4y-\frac{1}{5}y^5\right]_{-\sqrt{2}}^{\sqrt{2}}}{\frac{8\sqrt{2}}{3}} = \frac{\frac{16\sqrt{2}}{5}}{\frac{8\sqrt{2}}{3}} = \frac{6}{5}$$

$\overline{y} = 0$ (by symmetry)

16.



To find the intersection points, solve

$y^2 - 3y - 4 = -y$.

$y^2 - 2y - 4 = 0$

$$y = \frac{2 \pm \sqrt{20}}{2}$$

$y = 1 \pm \sqrt{5}$

$$\overline{x} = \frac{\frac{1}{2}\int_{1-\sqrt{5}}^{1+\sqrt{5}}\left[(-y)^2-(y^2-3y-4)^2\right]dy}{\int_{1-\sqrt{5}}^{1+\sqrt{5}}\left[(-y)-(y^2-3y-4)\right]dy}$$

$$= \frac{\frac{1}{2}\int_{1-\sqrt{5}}^{1+\sqrt{5}}(-y^4+6y^3-24y-16)dy}{\int_{1-\sqrt{5}}^{1+\sqrt{5}}(-y^2+2y+4)dy}$$

$$= \frac{\frac{1}{2}\left[-\frac{1}{5}y^5+\frac{3}{2}y^4-12y^2-16y\right]_{1-\sqrt{5}}^{1+\sqrt{5}}}{\left[-\frac{1}{3}y^3+y^2+4y\right]_{1-\sqrt{5}}^{1+\sqrt{5}}} = \frac{-20\sqrt{5}}{\frac{20\sqrt{5}}{3}}$$

$= -3$

$$\overline{y} = \frac{\int_{1-\sqrt{5}}^{1+\sqrt{5}} y\left[(-y)-(y^2-3y-4)\right]dy}{\int_{1-\sqrt{5}}^{1+\sqrt{5}}\left[(-y)-(y^2-3y-4)\right]dy}$$

$$= \frac{\int_{1-\sqrt{5}}^{1+\sqrt{5}}(-y^3+2y^2+4y)dy}{\frac{20\sqrt{5}}{3}}$$

$$= \frac{\left[-\frac{1}{4}y^4+\frac{2}{3}y^3+2y^2\right]_{1-\sqrt{5}}^{1+\sqrt{5}}}{\frac{20\sqrt{5}}{3}} = \frac{\frac{20\sqrt{5}}{3}}{\frac{20\sqrt{5}}{3}} = 1$$

17. We let δ be the density of the regions and A_i be the area of region i.

Region R_1:

$$m(R_1) = \delta A_1 = \delta(1/2)(1)(1) = \frac{1}{2}\delta$$

$$\overline{x}_1 = \frac{\int_0^1 x(x)dx}{\int_0^1 xdx} = \frac{\frac{1}{3}x^3\Big|_0^1}{\frac{1}{2}x^2\Big|_0^1} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Since R_1 is symmetric about the line $y = 1 - x$, the centroid must lie on this line. Therefore,

$\overline{y}_1 = 1 - \overline{x}_1 = 1 - \frac{2}{3} = \frac{1}{3}$; and we have

$$M_y(R_1) = \overline{x}_2 \cdot m(R_1) = \frac{1}{3}\delta$$

$$M_x(R_1) = \overline{y}_2 \cdot m(R_1) = \frac{1}{6}\delta$$

Region R_2:

$m(R_2) = \delta A_2 = \delta(2)(1) = 2\delta$

By symmetry we get

$\overline{x}_2 = 2$ and $\overline{y}_2 = \frac{1}{2}$.

Thus,

$M_y(R_2) = \overline{x}_2 \cdot m(R_2) = 4\delta$

$M_x(R_2) = \overline{y}_2 \cdot m(R_2) = \delta$

18. We can obtain the mass and moments for the whole region by adding the individual regions. Using the results from Problem 17 we get that

$$m = m(R_1) + m(R_2) = \frac{1}{2}\delta + 2\delta = \frac{5}{2}\delta$$

$$M_y = M_y(R_1) + M_y(R_2) = \frac{1}{3}\delta + 4\delta = \frac{13}{3}\delta$$

$$M_x = M_x(R_1) + M_x(R_2) = \frac{1}{6}\delta + \delta = \frac{7}{6}\delta$$

Therefore, the centroid is given by

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{13}{3}\delta}{\frac{5}{2}\delta} = \frac{26}{15}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{7}{6}\delta}{\frac{5}{2}\delta} = \frac{7}{15}$$

19. $m(R_1) = \delta\int_a^b (g(x) - f(x))dx$

$$m(R_2) = \delta\int_b^c (g(x) - f(x))dx$$

$$M_x(R_1) = \frac{\delta}{2}\int_a^b ((g(x))^2 - (f(x))^2)dx$$

$$M_x(R_2) = \frac{\delta}{2}\int_b^c ((g(x))^2 - (f(x))^2)dx$$

$$M_y(R_1) = \delta\int_a^b x(g(x) - f(x))dx$$

$$M_y(R_2) = \delta\int_b^c x(g(x) - f(x))dx$$

Now,

$$\begin{aligned} m(R_3) &= \delta\int_a^c (g(x) - f(x))dx \\ &= \delta\int_a^b (g(x) - f(x))dx + \delta\int_b^c (g(x) - f(x))dx \\ &= m(R_1) + m(R_2) \end{aligned}$$

$$\begin{aligned} M_x(R_3) &= \frac{\delta}{2}\int_a^c ((g(x))^2 - (f(x))^2)dx \\ &= \frac{\delta}{2}\int_a^b ((g(x))^2 - (f(x))^2)dx \\ &\quad + \frac{\delta}{2}\int_b^c ((g(x))^2 - (f(x))^2)dx \\ &= M_x(R_1) + M_x(R_2) \end{aligned}$$

$$\begin{aligned} M_y(R_3) &= \delta\int_a^c x(g(x) - f(x))dx \\ &= \delta\int_a^b x(g(x) - f(x))dx \\ &\quad + \delta\int_b^c x(g(x) - f(x))dx \\ &= M_y(R_1) + M_y(R_2) \end{aligned}$$

20. $m(R_1) = \delta\int_a^b (h(x) - g(x))dx$

$$m(R_2) = \delta\int_a^b (g(x) - f(x))dx$$

$$M_x(R_1) = \frac{\delta}{2}\int_a^b ((h(x))^2 - (g(x))^2)dx$$

$$M_x(R_2) = \frac{\delta}{2}\int_a^b ((g(x))^2 - (f(x))^2)dx$$

$$M_y(R_1) = \delta\int_a^b x(h(x) - g(x))dx$$

$$M_y(R_2) = \delta\int_a^b x(g(x) - f(x))dx$$

Now,

$$\begin{aligned} m(R_3) &= \delta\int_a^b (h(x) - f(x))dx \\ &= \delta\int_a^b (h(x) - g(x) + g(x) - f(x))dx \\ &= \delta\int_a^b (h(x) - g(x))dx + \delta\int_a^b (g(x) - f(x))dx \\ &= m(R_1) + m(R_2) \end{aligned}$$

$$\begin{aligned} M_x(R_3) &= \frac{\delta}{2}\int_a^b ((h(x))^2 - (f(x))^2)dx \\ &= \frac{\delta}{2}\int_a^b ((h(x))^2 - (g(x))^2 + (g(x))^2 - (f(x))^2)dx \\ &= \frac{\delta}{2}\int_a^b ((h(x))^2 - (g(x))^2)dx \\ &\quad + \frac{\delta}{2}\int_a^b ((g(x))^2 - (f(x))^2)dx \\ &= M_x(R_1) + M_x(R_2) \end{aligned}$$

$$\begin{aligned} M_y(R_3) &= \delta\int_a^b x(h(x) - f(x))dx \\ &= \delta\int_a^b x(h(x) - g(x) + g(x) - f(x))dx \\ &= \delta\int_a^b x(h(x) - g(x))dx + \delta\int_a^b x(g(x) - f(x))dx \\ &= M_y(R_1) + M_y(R_2) \end{aligned}$$

21. Let region 1 be the region bounded by $x = -2$, $x = 2$, $y = 0$, and $y = 1$, so $m_1 = 4 \cdot 1 = 4$.

By symmetry, $\bar{x}_1 = 0$ and $\bar{y}_1 = \frac{1}{2}$. Therefore $M_{1y} = \bar{x}_1 m_1 = 0$ and $M_{1x} = \bar{y}_1 m_1 = 2$.

Let region 2 be the region bounded by $x = -2$, $x = 1$, $y = -1$, and $y = 0$, so $m_2 = 3 \cdot 1 = 3$.

By symmetry, $\bar{x}_2 = -\frac{1}{2}$ and $\bar{y}_2 = -\frac{1}{2}$. Therefore $M_{2y} = \bar{x}_2 m_2 = -\frac{3}{2}$ and $M_{2x} = \bar{y}_2 m_2 = -\frac{3}{2}$.

$$\bar{x} = \frac{M_{1y} + M_{2y}}{m_1 + m_2} = \frac{-\frac{3}{2}}{7} = -\frac{3}{14}$$

$$\bar{y} = \frac{M_{1x} + M_{2x}}{m_1 + m_2} = \frac{\frac{1}{2}}{7} = \frac{1}{14}$$

22. Let region 1 be the region bounded by $x = –3$, $x = 1$, $y = –1$, and $y = 4$, so $m_1 = 20$. By symmetry, $\bar{x} = -1$ and $\bar{y}_1 = \frac{3}{2}$. Therefore, $M_{1y} = \bar{x}_1 m_1 = -20$ and $M_{1x} = \bar{y}_1 m_1 = 30$. Let region 2 be the region bounded by $x = –3$, $x = –2$, $y = –3$, and $y = –1$, so $m_2 = 2$. By symmetry, $\bar{x}_2 = -\frac{5}{2}$ and $\bar{y}_2 = -2$. Therefore, $M_{2y} = \bar{x}_2 m_2 = -5$ and $M_{2x} = \bar{y}_2 m_2 = -4$. Let region 3 be the region bounded by $x = 0$, $x = 1$, $y = –2$, and $y = –1$, so $m_3 = 1$. By symmetry, $\bar{x}_3 = \frac{1}{2}$ and $\bar{y}_3 = -\frac{3}{2}$. Therefore,

$M_{3y} = \bar{x}_3 m_3 = \frac{1}{2}$ and $M_{3x} = \bar{y}_3 m_3 = -\frac{3}{2}$.

$$\bar{x} = \frac{M_{1y} + M_{2y} + M_{3y}}{m_1 + m_2 + m_3} = \frac{-\frac{49}{2}}{23} = -\frac{49}{46}$$

$$\bar{y} = \frac{M_{1x} + M_{2x} + M_{3x}}{m_1 + m_2 + m_3} = \frac{\frac{49}{2}}{23} = \frac{49}{46}$$

23. Let region 1 be the region bounded by $x = –2$, $x = 2$, $y = 2$, and $y = 4$, so $m_1 = 4 \cdot 2 = 8$. By symmetry, $\bar{x}_1 = 0$ and $\bar{y}_1 = 3$. Therefore, $M_{1y} = \bar{x}_1 m_1 = 0$ and $M_{1x} = \bar{y}_1 m_1 = 24$. Let region 2 be the region bounded by $x = –1$, $x = 2$, $y = 0$, and $y = 2$, so $m_2 = 3 \cdot 2 = 6$. By symmetry, $\bar{x}_2 = \frac{1}{2}$ and $\bar{y}_2 = 1$. Therefore, $M_{2y} = \bar{x}_2 m_2 = 3$ and $M_{2x} = \bar{y}_2 m_2 = 6$. Let region 3 be the region bounded by $x = 2$, $x = 4$, $y = 0$, and $y = 1$, so $m_3 = 2 \cdot 1 = 2$. By symmetry, $\bar{x}_3 = 3$ and $\bar{y}_2 = \frac{1}{2}$. Therefore, $M_{3y} = \bar{x}_3 m_3 = 6$ and $M_{3x} = \bar{y}_3 m_3 = 1$.

$$\bar{x} = \frac{M_{1y} + M_{2y} + M_{3y}}{m_1 + m_2 + m_3} = \frac{9}{16}$$

$$\bar{y} = \frac{M_{1x} + M_{2x} + M_{3x}}{m_1 + m_2 + m_3} = \frac{31}{16}$$

24. Let region 1 be the region bounded by $x = –3$, $x = –1$, $y = –2$, and $y = 1$, so $m_1 = 6$. By symmetry, $\bar{x}_1 = -2$ and $\bar{y}_1 = -\frac{1}{2}$. Therefore, $M_{1y} = \bar{x}_1 m_1 = -12$ and $M_{1x} = \bar{y}_1 m_1 = -3$. Let region 2 be the region bounded by $x = –1$, $x = 0$, $y = –2$, and $y = 0$, so $m_2 = 2$. By symmetry, $\bar{x}_2 = -\frac{1}{2}$ and $\bar{y}_2 = -1$. Therefore, $M_{2y} = \bar{x}_2 m_2 = -1$ and $M_{2x} = \bar{y}_2 m_2 = -2$. Let region 3 be the remaining region, so $m_3 = 22$. By symmetry, $\bar{x}_3 = 2$ and $\bar{y}_3 = -\frac{1}{2}$. Therefore, $M_{3y} = \bar{x}_3 m_3 = 44$ and $M_{3x} = \bar{y}_3 m_3 = -11$.

$$\bar{x} = \frac{M_{1y} + M_{2y} + M_{3y}}{m_1 + m_2 + m_3} = \frac{31}{30}$$

$$\bar{y} = \frac{M_{1x} + M_{2x} + M_{3x}}{m_1 + m_2 + m_3} = -\frac{16}{30} = -\frac{8}{15}$$

25. $A = \int_0^1 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}$

From Problem 11, $\bar{x} = \frac{4}{5}$.

$$V = A(2\pi\bar{x}) = \frac{1}{4}\left(2\pi \cdot \frac{4}{5}\right) = \frac{2\pi}{5}$$

Using cylindrical shells:

$$V = 2\pi\int_0^1 x \cdot x^3 dx = 2\pi\int_0^1 x^4 dx = 2\pi\left[\frac{1}{5}x^5\right]_0^1 = \frac{2\pi}{5}$$

26. The area of the region is πa^2. The centroid is the center (0, 0) of the circle. It travels a distance of $2\pi(2a) = 4\pi a$. $V = 4\pi^2 a^3$

27. The volume of a sphere of radius a is $\frac{4}{3}\pi a^3$. If the semicircle $y = \sqrt{a^2 - x^2}$ is revolved about the x-axis the result is a sphere of radius a. The centroid of the region travels a distance of $2\pi\bar{y}$. The area of the region is $\frac{1}{2}\pi a^2$. Pappus's Theorem says that

$$(2\pi\bar{y})\left(\frac{1}{2}\pi a^2\right) = \pi^2 a^2 \bar{y} = \frac{4}{3}\pi a^3.$$

$\bar{y} = \frac{4a}{3\pi}$, $\bar{x} = 0$ (by symmetry)

28. Consider a slice at x rotated about the y-axis.

$\Delta V = 2\pi x h(x)\Delta x$, so $V = 2\pi\int_a^b xh(x)dx$.

$\Delta m \approx h(x)\Delta x$, so $m = \int_a^b h(x)dx = A$.

$\Delta M_y \approx xh(x)\Delta x$, so $M_y = \int_a^b xh(x)dx$.

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b xh(x)dx}{A}$$

The distance traveled by the centroid is $2\pi\bar{x}$.

$(2\pi\bar{x})A = 2\pi\int_a^b xh(x)dx$

Therefore, $V = 2\pi\bar{x}A$.

29. a. $\Delta V \approx 2\pi(K-y)w(y)\Delta y$

$V = 2\pi\int_c^d (K-y)w(y)dy$

b. $\Delta m \approx w(y)\Delta y$, so $m = \int_c^d w(y)dy = A$.

$\Delta M_x \approx yw(y)\Delta y$, so $M_x = \int_c^d yw(y)dy$.

$\overline{y} = \dfrac{\int_c^d yw(y)dy}{A}$

The distance traveled by the centroid is $2\pi(K-\overline{y})$.

$2\pi(K-\overline{y})A = 2\pi(KA - M_x)$

$= 2\pi\left(\int_c^d Kw(y)dy - \int_c^d yw(y)dy\right)$

$= 2\pi\int_c^d (K-y)w(y)dy$

Therefore, $V = 2\pi(K-\overline{y})A$.

30. a. $m = \dfrac{1}{2}bh$

The length of a segment at y is $b - \dfrac{b}{h}y$.

$\Delta M_x \approx y\left(b - \dfrac{b}{h}y\right)\Delta y = \left(by - \dfrac{b}{h}y^2\right)\Delta y$

$M_x = \int_0^h \left(by - \dfrac{b}{h}y^2\right)dy$

$= \left[\dfrac{1}{2}by^2 - \dfrac{b}{3h}y^3\right]_0^h = \dfrac{1}{6}bh^2$

$\overline{y} = \dfrac{M_x}{m} = \dfrac{h}{3}$

b. $A = \dfrac{1}{2}bh$; the distance traveled by the centroid is $2\pi\left(k - \dfrac{h}{3}\right)$.

$V = 2\pi\left(k - \dfrac{h}{3}\right)\left(\dfrac{1}{2}bh\right) = \dfrac{\pi bh}{3}(3k-h)$

31. a. The area of a regular polygon P of $2n$ sides is $2r^2 n\sin\dfrac{\pi}{2n}\cos\dfrac{\pi}{2n}$. (To find this consider the isosceles triangles with one vertex at the center of the polygon and the other vertices on adjacent corners of the polygon. Each such triangle has base of length $2r\sin\dfrac{\pi}{2n}$ and height $r\cos\dfrac{\pi}{2n}$.) Since P is a regular polygon the centroid is at its center. The distance from the centroid to any side is $r\cos\dfrac{\pi}{2n}$, so the centroid travels a distance of $2\pi r\cos\dfrac{\pi}{2n}$.

Thus, by Pappus's Theorem, the volume of the resulting solid is

$\left(2\pi r\cos\dfrac{\pi}{2n}\right)\left(2r^2 n\sin\dfrac{\pi}{2n}\cos\dfrac{\pi}{2n}\right)$

$= 4\pi r^3 n\sin\dfrac{\pi}{2n}\cos^2\dfrac{\pi}{2n}$.

b. $\lim_{n\to\infty} 4\pi r^3 n\sin\dfrac{\pi}{2n}\cos^2\dfrac{\pi}{2n}$

$\lim_{n\to\infty} \dfrac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}} 2\pi^2 r^3\cos^2\dfrac{\pi}{2n} = 2\pi^2 r^3$

As $n\to\infty$, the regular polygon approaches a circle. Using Pappus's Theorem on the circle of area πr^2 whose centroid (= center) travels a distance of $2\pi r$, the volume of the solid is $(\pi r^2)(2\pi r) = 2\pi^2 r^3$ which agrees with the results from the polygon.

32. a. The graph of $f(\sin x)$ on [0, π] is symmetric about the line $x = \dfrac{\pi}{2}$ since $f(\sin x) = f(\sin(\pi - x))$. Thus $\overline{x} = \dfrac{\pi}{2}$.

$\overline{x} = \dfrac{\int_0^\pi x f(\sin x)\,dx}{\int_0^\pi f(\sin x)dx} = \dfrac{\pi}{2}$

Therefore

$\int_0^\pi x f(\sin x)dx = \dfrac{\pi}{2}\int_0^\pi f(\sin x)dx$

b. $\sin x\cos^4 x = \sin x(1-\sin^2 x)^2$, so $f(x) = x(1-x^2)^2$.

$\int_0^\pi x\sin x\cos^4 x\,dx = \dfrac{\pi}{2}\int_0^\pi \sin x\cos^4 x\,dx$

$= \dfrac{\pi}{2}\left[-\dfrac{1}{5}\cos^5 x\right]_0^\pi = \dfrac{\pi}{5}$

33. Consider the region $S - R$.

$$\bar{y}_{S-R} = \frac{\frac{1}{2}\int_0^1 \left[g^2(x) - f^2(x)\right]dx}{S-R} \ge \bar{y}_R$$

$$= \frac{\frac{1}{2}\int_0^1 f^2(x)dx}{R}$$

$$\frac{1}{2}R\int_0^1 \left[g^2(x) - f^2(x)\right]dx \ge \frac{1}{2}(S-R)\int_0^1 f^2(x)dx$$

$$\frac{1}{2}R\int_0^1 \left[g^2(x) - f^2(x)\right]dx + \frac{1}{2}R\int_0^1 f^2(x)dx$$

$$\ge \frac{1}{2}(S-R)\int_0^1 f^2(x)dx + \frac{1}{2}R\int_0^1 f^2(x)dx$$

$$\frac{1}{2}R\int_0^1 g^2(x)dx \ge \frac{1}{2}S\int_0^1 f^2(x)dx$$

$$\frac{\frac{1}{2}\int_0^1 g^2(x)\,dx}{S} \ge \frac{\frac{1}{2}\int_0^1 f^2(x)dx}{R}$$

$$\bar{y}_S \ge \bar{y}_R$$

34. To approximate the centroid, we can lay the figure on the x-axis (flat side down) and put the shortest side against the y-axis. Next we can use the eight regions between measurements to approximate the centroid. We will let h_i ,the height of the *i*th region, be approximated by the height at the right end of the interval. Each interval is of width $\Delta x = 5$ cm. The centroid can be approximated as

$$\bar{x} \approx \frac{\sum_{i=1}^{8} x_i h_i}{\sum_{i=1}^{8} h_i} = \frac{(5)(6.5) + (10)(8) + \cdots + (35)(10) + (40)(8)}{6.5 + 8 + \cdots + 10 + 8}$$

$$= \frac{1695}{72.5} \approx 23.38$$

$$\bar{y} \approx \frac{\frac{1}{2}\sum_{i=1}^{8}(h_i)^2}{\sum_{i=1}^{8} h_i} = \frac{(1/2)(6.5^2 + 8^2 + \cdots + 10^2 + 8^2)}{(6.5 + 8 + \cdots + 10 + 8)}$$

$$= \frac{335.875}{72.5} \approx 4.63$$

35. First we place the lamina so that the origin is centered inside the hole. We then recompute the centroid of Problem 34 (in this position) as

$$\bar{x} \approx \frac{\sum_{i=1}^{8} x_i h_i}{\sum_{i=1}^{8} h_i}$$

$$= \frac{(-25)(6.5) + (-15)(8) + \cdots + (5)(10) + (10)(8)}{6.5 + 8 + \cdots + 10 + 8}$$

$$= \frac{-480}{72.5} \approx -6.62$$

$$\bar{y} \approx \frac{\frac{1}{2}\sum_{i=1}^{8}((h_i - 4)^2 - (-4)^2)}{\sum_{i=1}^{8} h_i}$$

$$= \frac{(1/2)((2.5^2 - (-4)^2) + \cdots + (4^2 - (-4)^2))}{6.5 + 8 + \cdots + 10 + 8}$$

$$= \frac{45.875}{72.5} \approx 0.633$$

A quick computation will show that these values agree with those in Problem 34 (using a different reference point).

Now consider the whole lamina as R_3 , the circular hole as R_2 , and the remaining lamina as R_1 . We can find the centroid of R_1 by noting that

$M_x(R_1) = M_x(R_3) - M_x(R_2)$

and similarly for $M_y(R_1)$.

From symmetry, we know that the centroid of a circle is at the center. Therefore, both $M_x(R_2)$ and $M_y(R_2)$ must be zero in our case.

This leads to the following equations

$$\bar{x} = \frac{M_y(R_3) - M_y(R_2)}{m(R_3) - m(R_2)}$$

$$= \frac{\delta\Delta x(-480)}{\delta\Delta x(72.5) - \delta\pi(2.5)^2}$$

$$= \frac{-2400}{342.87} \approx -7$$

$$\bar{y} = \frac{M_x(R_3) - M_x(R_2)}{m(R_3) - m(R_2)}$$

$$= \frac{\delta\Delta x(45.875)}{\delta\Delta x(72.5) - \delta\pi(2.5)^2}$$

$$= \frac{229.375}{342.87} \approx 0.669$$

Thus, the centroid is 7 cm above the center of the hole and 0.669 cm to the right of the center of the hole.

36. This problem is much like Problem 34 except we don't have one side that is completely flat. In this problem, it will be necessary, in some regions, to find the value of $g(x)$ instead of just $f(x) - g(x)$. We will use the 19 regions in the figure to approximate the centroid. Again we choose the height of a region to be approximately the value at the right end of that region. Each region has a width of 20 miles. We will place the north-east corner of the state at the origin.
The centroid is approximately

$$\bar{x} \approx \frac{\sum_{i=1}^{19} x_i(f(x_i) - g(x_i))}{\sum_{i=1}^{19}(f(x_i) - g(x_i))}$$

$$= \frac{(20)(145-13) + (40)(149-10) + \cdots (380)(85-85)}{(145-13) + (149-19) + \cdots (85-85)}$$

$$= \frac{482,860}{2780} \approx 173.69$$

$$\bar{y} \approx \frac{\frac{1}{2}\sum_{i=1}^{19}[(f(x_i))^2 - (g(x_i))^2]}{\sum_{i=1}^{19}(f(x_i) - g(x_i))}$$

$$= \frac{\frac{1}{2}\left[(145^2 - 13^2) + (149^2 - 10^2) + \cdots + (85^2 - 85^2)\right]}{(145-13) + (149-19) + \cdots + (85-85)}$$

$$= \frac{230,805}{2780} \approx 83.02$$

This would put the geographic center of Illinois just south-east of Lincoln, IL.

5.7 Concepts Review

1. discrete, continuous

2. sum, integral

3. $\int_0^5 f(x)\,dx$

4. cumulative distribution function

Problem Set 5.7

1. a. $P(X \geq 2) = P(2) + P(3) = 0.05 + 0.05 = 0.1$

b. $E(X) = \sum_{i=1}^{4} x_i p_i$
$= 0(0.8) + 1(0.1) + 2(0.05) + 3(0.05)$
$= 0.35$

2. a. $P(X \geq 2) = P(2) + P(3) + P(4)$
$= 0.05 + 0.05 + 0.05 = 0.15$

b. $E(X) = \sum_{i=1}^{5} x_i p_i$
$= 0(0.7) + 1(0.15) + 2(0.05)$
$+ 3(0.05) + 4(0.5)$
$= 0.6$

3. a. $P(X \geq 2) = P(2) = 0.2$

b. $E(X) = -2(0.2) + (-1)(0.2) + 0(0.2)$
$+ 1(0.2) + 2(0.2)$
$= 0$

4. a. $P(X \geq 2) = P(2) = 0.1$

b. $E(X) = -2(0.1) + (-1)(0.2) + 0(0.4)$
$+ 1(0.2) + 2(0.1)$
$= 0$

5. a. $P(X \geq 2) = P(2) + P(3) + P(4)$
$= 0.2 + 0.2 + 0.2$
$= 0.6$

b. $E(X) = 1(0.4) + 2(0.2) + 3(0.2) + 4(0.2)$
$= 2.2$

6. a. $P(X \geq 2) = P(100) + P(1000)$
$= 0.018 + 0.002 = 0.02$

b. $E(X) = -0.1(0.98) + 100(0.018)$
$+ 1000(0.002)$
$= 3.702$

7. a. $P(X \geq 2) = P(2) + P(3) + P(4)$
$= \frac{3}{10} + \frac{2}{10} + \frac{1}{10} = \frac{6}{10} = 0.6$

b. $E(X) = 1(0.4) + 2(0.3) + 3(0.2) + 4(0.1) = 2$

8. **a.** $P(X \geq 2) = P(2) + P(3) + P(4)$

$$= \frac{0^2}{10} + \frac{(-1)^2}{10} + \frac{(-2)^2}{10} = \frac{5}{10} = 0.5$$

b. $E(X) = 0(0.4) + 1(0.1) + 2(0) + 3(0.1) + 4(0.4)$
$= 2$

9. **a.** $P(X \geq 2) = \int_2^{20} \frac{1}{20}\,dx = \frac{1}{20} \cdot 18 = 0.9$

b. $E(X) = \int_0^{20} x \cdot \frac{1}{20}\,dx = \left[\frac{x^2}{40}\right]_0^{20} = 10$

c. For x between 0 and 20,

$$F(x) = \int_0^x \frac{1}{20}\,dt = \frac{1}{20} \cdot x = \frac{x}{20}$$

10. **a.** $P(X \geq 2) = \int_2^{20} \frac{1}{40}\,dx = \frac{1}{40} \cdot 18 = 0.45$

b. $E(X) = \int_{-20}^{20} x \cdot \frac{1}{40}\,dx = \left[\frac{x^2}{80}\right]_{-20}^{20} = 5 - 5 = 0$

c. For $-20 \leq x \leq 20$,

$$F(x) = \int_{-20}^x \frac{1}{40}\,dt = \frac{1}{40}(x+20) = \frac{1}{40}x + \frac{1}{2}$$

11. **a.** $P(X \geq 2) = \int_2^8 \frac{3}{256} x(8-x)\,dx$

$$= \frac{3}{256}\left[4x^2 - \frac{x^3}{3}\right]_2^8 = \frac{3}{256} \cdot 72 = \frac{27}{32}$$

b. $E(X) = \int_0^8 x \cdot \frac{3}{256} x(8-x)\,dx$

$$= \frac{3}{256}\int_0^8 \left(8x^2 - x^3\right)dx$$

$$= \frac{3}{256}\left[\frac{8x^3}{3} - \frac{x^4}{4}\right]_0^8 = 4$$

c. For $0 \leq x \leq 8$

$$F(x) = \int_0^x \frac{3}{256} t(8-t)\,dt = \frac{3}{256}\left[4t^2 - \frac{t^3}{3}\right]_0^x$$

$$= \frac{3}{64}x^2 - \frac{1}{256}x^3$$

12. **a.** $P(X \geq 2) = \int_2^{20} \frac{3}{4000} x(20-x)\,dx$

$$= \frac{3}{4000}\left[10x^2 - \frac{x^3}{3}\right]_2^{20} = 0.972$$

b. $E(X) = \int_0^{20} x \cdot \frac{3}{4000} x(20-x)\,dx$

$$= \frac{3}{4000}\int_0^{20}\left(20x^2 - x^3\right)dx$$

$$= \frac{3}{4000}\left[\frac{20x^3}{3} - \frac{x^4}{4}\right]_0^{20} = 10$$

c. For $0 \leq x \leq 20$

$$F(x) = \int_0^x \frac{3}{4000} t(20-t)\,dt$$

$$= \frac{3}{4000}\left[10t^2 - \frac{t^3}{3}\right]_0^x = \frac{3}{400}x^2 - \frac{1}{4000}x^3$$

13. **a.** $P(X \geq 2) = \int_2^4 \frac{3}{64} x^2(4-x)\,dx$

$$= \frac{3}{64}\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_2^4 = 0.6875$$

b. $E(X) = \int_0^4 x \cdot \frac{3}{64} x^2(4-x)\,dx$

$$= \frac{3}{64}\int_0^4\left(4x^3 - x^4\right)dx = \frac{3}{64}\left[x^4 - \frac{x^5}{5}\right]_0^4 = 2.4$$

c. For $0 \leq x \leq 4$

$$F(x) = \int_0^x \frac{3}{64} t^2(4-t)\,dt = \frac{3}{64}\left[\frac{4t^3}{3} - \frac{t^4}{4}\right]_0^x$$

$$= \frac{1}{16}x^3 - \frac{3}{256}x^4$$

14. **a.** $P(X \geq 2) = \int_2^8 \frac{1}{32}(8-x)\,dx$

$$= \frac{1}{32}\left[8x - \frac{x^2}{2}\right]_2^8 = \frac{9}{16}$$

b. $E(X) = \int_0^8 x \cdot \frac{1}{32}(8-x)\,dx$

$$= \frac{1}{32}\left[4x^2 - \frac{x^3}{3}\right]_0^8 = \frac{8}{3}$$

c. For $0 \leq x \leq 8$

$$F(x) = \int_0^x \frac{1}{32}(8-t)\,dt = \frac{1}{32}\left[8t - \frac{t^2}{2}\right]_0^x$$

$$= \frac{1}{4}x - \frac{1}{64}x^2$$

15. **a.** $P(X \ge 2) = \int_2^4 \frac{\pi}{8} \sin\left(\frac{\pi x}{4}\right) dx$

$= \frac{\pi}{8}\left[-\frac{4}{\pi}\cos\frac{\pi x}{4}\right]_2^4 = -\frac{1}{2}(-1-0) = \frac{1}{2}$

b. $E(X) = \int_0^4 x \cdot \frac{\pi}{8} \sin\left(\frac{\pi x}{4}\right) dx$

Using integration by parts or a CAS, $E(X) = 2$.

c. For $0 \le x \le 4$

$F(x) = \int_0^x \frac{\pi}{8} \sin\left(\frac{\pi t}{4}\right) dt = \frac{\pi}{8}\left[\frac{-4}{\pi}\cos\frac{\pi t}{4}\right]_0^x$

$= -\frac{1}{2}\left(\cos\frac{\pi x}{4} - 1\right) = -\frac{1}{2}\cos\frac{\pi x}{4} + \frac{1}{2}$

16. **a.** $P(X \ge 2) = \int_2^4 \frac{\pi}{8} \cos\left(\frac{\pi x}{8}\right) dx$

$= \left[\sin\left(\frac{\pi x}{8}\right)\right]_2^4 = \sin\frac{\pi}{2} - \sin\frac{\pi}{4} = 1 - \frac{1}{\sqrt{2}}$

b. $E(X) = \int_0^4 x \cdot \frac{\pi}{8} \cos\left(\frac{\pi x}{8}\right) dx$

Using a CAS, $E(X) \approx 1.4535$

c. For $0 \le x \le 4$

$F(x) = \int_0^x \frac{\pi}{8} \cos\left(\frac{\pi t}{8}\right) dt = \left[\sin\left(\frac{\pi t}{8}\right)\right]_0^x$

$= \sin\left(\frac{\pi x}{8}\right)$

17. **a.** $P(X \ge 2) = \int_2^4 \frac{4}{3x^2} dx = \left[-\frac{4}{3x}\right]_2^4 = \frac{1}{3}$

b. $E(X) = \int_1^4 x \cdot \frac{4}{3x^2} dx = \left[\frac{4}{3}\ln x\right]_1^4$

$= \frac{4}{3}\ln 4 \approx 1.85$

c. For $1 \le x \le 4$

$F(x) = \int_1^x \frac{4}{3t^2} dt = \left[-\frac{4}{3t}\right]_1^1 = \frac{-4}{3x} + \frac{4}{3}$

$= \frac{4x-4}{3x}$

18. **a.** $P(X \ge 2) = \int_2^9 \frac{81}{40x^3} dx = \left[-\frac{81}{80x^2}\right]_2^9$

$= \frac{77}{320} \approx 0.24$

b. $E(X) = \int_1^9 x \cdot \frac{81}{40x^3} dx = \left[-\frac{81}{40x}\right]_1^9 = 1.8$

c. For $1 \le x \le 9$

$F(x) = \int_1^x \frac{81}{40t^3} dt = \left[-\frac{81}{80t^2}\right]_1^x$

$= -\frac{81}{80x^2} + \frac{81}{80} = \frac{81x^2 - 81}{80x^2}$

19. Proof of $F'(x) = f(x)$:

By definition, $F(x) = \int_A^x f(t)\,dt$. By the First Fundamental Theorem of Calculus, $F'(x) = f(x)$.

Proof of $F(A) = 0$ and $F(B) = 1$:

$F(A) = \int_A^A f(x)\,dx = 0;$

$F(B) = \int_A^B f(x)\,dx = 1$

Proof of $P(a \le X \le b) = F(b) - F(a)$:

$P(a \le X \le b) = \int_a^b f(x)\,dx = F(b) - F(a)$ due to the Second Fundamental Theorem of Calculus.

20. **a.** The midpoint of the interval [a,b] is $\frac{a+b}{2}$.

$P\left(X < \frac{a+b}{2}\right) = P\left(X \le \frac{a+b}{2}\right)$

$= \int_a^{\frac{a+b}{2}} \frac{1}{b-a} dx = \frac{1}{b-a}\left(\frac{a+b}{2} - a\right)$

$= \frac{1}{b-a} \cdot \frac{b-a}{2} = \frac{1}{2}$

b. $E(X) = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$

$= \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$

c. $F(x) = \int_a^x \frac{1}{b-a} dt = \frac{1}{b-a}(x-a) = \frac{x-a}{b-a}$

21. The median will be the solution to the equation $\int_a^{x_0} \frac{1}{b-a} dx = 0.5$.

$\frac{1}{b-a}(x_0 - a) = 0.5$

$x_0 - a = \frac{b-a}{2}$

$x_0 = \frac{a+b}{2}$

22. The graph of $f(x)=\frac{15}{512}x^2(4-x)^2$ is symmetric about the line x = 2. Consequently, $P(X \le 2)=0.5$ and 2 must be the median of *X*.

23. Since the PDF must integrate to one, solve $\int_0^5 kx(5-x)\,dx=1$.

$$\left[\frac{5kx^2}{2}-\frac{kx^3}{3}\right]_0^5=1$$

$$\frac{125k}{2}-\frac{125k}{3}=1$$

$$375k-250k=6$$

$$k=\frac{6}{125}$$

24. Solve $\int_0^5 \mathrm{kx}^2(5-x)^2\,dx=1$

$$k\int_0^5\left(25x^2-10x^3+x^4\right)dx=1$$

$$k\left[\frac{25x^3}{3}-\frac{5x^4}{2}+\frac{x^5}{5}\right]_0^5=1$$

$$\frac{625}{6}k=1$$

$$k=\frac{6}{625}$$

25. **a.** Solve $\int_0^4 k\left(2-|x-2|\right)dx=1$

Due to the symmetry about the line x = 2, the solution can be found by solving

$$2\int_0^2 kx\,dx=1$$

$$k\cdot x^2\Big|_0^2=1$$

$$4k=1$$

$$k=\frac{1}{4}$$

b. $P(3\le X\le 4)=\int_3^4\frac{1}{4}\left(2-|x-2|\right)dx$

$$=\int_3^4\frac{1}{4}(2-(x-2))\,dx=\frac{1}{4}\int_3^4(4-x)\,dx$$

$$=\frac{1}{4}\left[4x-\frac{x^2}{2}\right]_3^4=\frac{1}{8}$$

c. $E(X)=\int_0^4 x\cdot\frac{1}{4}(2-|x-2|)\,dx$

$$=\int_0^2 x\cdot\frac{1}{4}(2+(x-2))\,dx+\int_2^4 x\cdot\frac{1}{4}(2-(x-2))\,dx$$

$$=\frac{1}{4}\int_0^2 x^2\,dx+\frac{1}{4}\int_2^4(4x-x^2)\,dx$$

$$=\frac{1}{12}x^3\Big|_0^2+\frac{1}{4}\left[2x^2-\frac{x^3}{3}\right]_2^4=\frac{2}{3}+\frac{4}{3}=2$$

d. If $0\le x\le 2$, $F(x)=\int_0^x\frac{1}{4}t\,dt=\left[\frac{t^2}{8}\right]_0^x=\frac{x^2}{8}$

If $2<x\le 4$, $F(x)=\int_0^2\frac{1}{4}x\,dx+\int_2^x\frac{1}{4}(4-t)\,dt$

$$=\frac{x^2}{8}\Big|_0^2+\frac{1}{4}\left[4t-\frac{t^2}{2}\right]_2^x=\frac{1}{2}+\left(x-\frac{x^2}{8}-\frac{3}{2}\right)$$

$$=-\frac{x^2}{8}+x-1$$

$$F(x)=\begin{cases}0 & \text{if } x<0\\ \frac{x^2}{8} & \text{if } 0\le x\le 2\\ -\frac{x^2}{8}+x-1 & \text{if } 2<x\le 4\\ 1 & \text{if } x>4\end{cases}$$

e. Using a similar procedure as shown in part (a), the PDF for *Y* is

$$f(y)=\frac{1}{14,400}\left(120-|y-120|\right)$$

If $0\le y<120$, $F(y)=\int_0^y\frac{1}{14,400}t\,dt$

$$=\left[\frac{t^2}{28,800}\right]_0^y=\frac{y^2}{28,800}$$

If $120<y\le 240$,

$$F(y)=\frac{1}{2}+\int_{120}^y\frac{1}{14,400}(240-t)\,dt$$

$$=\frac{1}{2}+\frac{1}{14,400}\left[240t-\frac{t^2}{2}\right]_{120}^y$$

$$=\frac{1}{2}+\frac{y}{60}-\frac{y^2}{28,800}-\frac{3}{2}=-\frac{y^2}{28,800}+\frac{y}{60}-1$$

$$F(x)=\begin{cases}0 & \text{if } y<0\\ \frac{y^2}{28,800} & \text{if } 0\le y\le 120\\ -\frac{y^2}{28,800}+\frac{y}{60}-1 & \text{if } 120<y\le 240\\ 1 & \text{if } y>240\end{cases}$$

26. a. Solve $\int_0^{180} kx^2(180-x)\,dx = 1.$

$$k\left[60x^3 - \frac{x^4}{4}\right]_0^{180} = 1$$

$$k = \frac{1}{87,480,000}$$

b. $P(100 \le X \le 150)$

$$= \int_{100}^{150} \frac{1}{87,480,000} x^2(180-x)\,dx$$

$$= \frac{1}{87,480,000}\left[60x^3 - \frac{x^4}{4}\right]_{100}^{150} \approx 0.468$$

c. $E(X) = \int_0^{180} x \cdot \frac{1}{87,480,000} x^2(180-x)\,dx$

$$= \frac{1}{87,480,000}\left[45x^4 - \frac{x^5}{5}\right]_0^{180} = 108$$

27. a. Solve $\int_0^{0.6} kx^6(0.6-x)^8\,dx = 1.$

$$k\int_0^{0.6} x^6(0.6-x)^8\,dx = 1$$

Using a CAS, k ≈ 95,802,719

b. The probability that a unit is scrapped is $1 - P(0.35 \le X \le 0.45)$

$$= 1 - k\int_{0.35}^{0.45} x^6(0.6-x)^8\,dx$$

≈ 0.884 using a CAS

c. $E(X) = \int_0^{0.6} x \cdot kx^6(0.6-x)^8\,dx$

$$= k\int_0^{0.6} x^7(0.6-x)^8\,dx$$

≈ 0.2625

d. $F(x) = \int_0^x 95,802,719t^6(0.6-t)^8\,dt$

Using a CAS,

$$F(x) \approx 6,386,850x^7(x^8 - 5.14286x^7 + 11.6308x^6 - 15.12x^5 + 12.3709x^4 - 6.53184x^3 + 2.17728x^2 - 0.419904x + 0.36)$$

e. If X = measurement in mm, and Y = measurement in inches, then $Y = X/25.4$. Thus,

$$F_Y(y) = P(Y \le y) = P(X/25.4 \le y) = P(X \le 25.4y) = F(25.4y)$$

where $F(x)$ is given in part (d).

Alternatively, we can proceed as follows:

Solve $\int_0^{3/127} k \cdot y^6\left(\frac{3}{127} - y\right)^8 dy = 1$ using a CAS.

$k \approx 1.132096857 \times 10^{29}$

$$F_Y(y) = \int_0^y k \cdot t^6\left(\frac{3}{127} - t\right)^8 dt$$

Using a CAS,

$$F_Y(y) \approx (7.54731\times10^{27})y^7(y^8 - 0.202475y^7 + 0.01802y^6 - 0.000923y^5 + 0.00003y^4 - (6.17827\times10^{-7})y^3 + (8.108\times10^{-9})y^2 - (6.156\times10^{-11})y + 2.07746\times10^{-13})$$

28. a. Solve $\int_0^{200} kx^2(200-x)^8\,dx = 1.$

Using a CAS, $k \approx 2.417 \times 10^{-23}$

b. The probability that a batch is not accepted is

$$P(X \ge 100) = k\int_{100}^{200} x^2(200-x)^8\,dx$$

≈ 0.0327 using a CAS.

c. $E(X) = k\int_0^{200} x \cdot x^2(200-x)^8\,dx$

= 50 using a CAS

d. $F(x) = \int_0^x (2.417\times10^{-23})t^2(200-t)^8\,dt$

Using a CAS, $F(x) \approx (2.19727\times10^{-24})x^3 \cdot$

$$(x^8 - 1760x^7 + 136889x^6 - (6.16\times10^8)x^5 + (1.76\times10^{11})x^4 - (3.2853\times10^{13})x^3 + (3.942\times10^{15})x^2 - (2.816\times10^{17})x + 9.39\times10^{18})$$

e. Solve $\int_0^{100} kx^2(100-x)^8\,dx.$ Using a CAS,

$k = 4.95\times10^{-20}$

$$F(x) = \int_0^y \left(4.95\times10^{-20}\right)t^2\left(100-t\right)^8 dt$$

Using a CAS,

$$F(x) \approx (4.5\times10^{-21})x^3 (x^8 - 880x^7 + 342,222x^6 - (7.7\times10^7)x^5 + (1.1\times10^{10})x^4 - (1.027\times10^{12})x^3 + (6.16\times10^{13})x^2 - (2.2\times10^{15})x + 3.667\times10^{16})$$

29. The PDF for the random variable X is

$$f(x)=\begin{cases}1 & \text{if } 0\le x\le 1\\ 0 & \text{otherwise}\end{cases}$$

From Problem 20, the CDF for X is $F(x)=x$

Y is the distance from $(1, X)$ to the origin, so

$$Y=\sqrt{(1-0)^2+(X-0)^2}=\sqrt{1+X^2}$$

Here we have a one-to-one transformation from the set $\{x: 0\le x\le 1\}$ to $\{y: 1\le y\le \sqrt{2}\}$. For every $1<a<b<\sqrt{2}$, the event $a<Y<b$ will occur when, and only when,

$\sqrt{a^2-1}<X<\sqrt{b^2-1}$.

If we let $a=1$ and $b=y$, we can obtain the CDF for Y.

$$\begin{aligned}P(1\le Y\le y)&=P\left(\sqrt{1^2-1}\le X\le\sqrt{y^2-1}\right)\\&=P\left(0\le X\le\sqrt{y^2-1}\right)\\&=F\left(\sqrt{y^2-1}\right)=\sqrt{y^2-1}\end{aligned}$$

To find the PDF, we differentiate the CDF with respect to y.

$$PDF=\frac{d}{dy}\sqrt{y^2-1}=\frac{1}{2}\cdot\frac{1}{\sqrt{y^2-1}}\cdot 2y=\frac{y}{\sqrt{y^2-1}}$$

Therefore, for $0\le y\le\sqrt{2}$ the PDF and CDF are respectively

$g(y)=\dfrac{y}{\sqrt{y^2-1}}$ and $G(y)=\sqrt{y^2-1}$.

30. $P(X=x)=\int_x^x f(t)\,dt=0.$ Consequently, $P(X<c)=P(X\le c).$ As a result, all four expressions, $P(a<X<b)$, $P(a\le X\le b)$, $P(a<X\le b)$ and $P(a\le X<b)$, are equivalent.

31. By the defintion of a complement of a set, $A\cup A^c=S$, where S denotes the sample space.

Since $P(S)=1$, $P(A\cup A^c)=1$.

Since $P(A\cup A^c)=P(A)+P(A^c)$,

$P(A)+P(A^c)=1$ and $P(A^c)=1-P(A)$.

32. $P(X\ge 1)=1-P(X<1)$

For Problem 1, $1-P(X<1)=1-P(X=0)$

$=1-0.8=0.2$

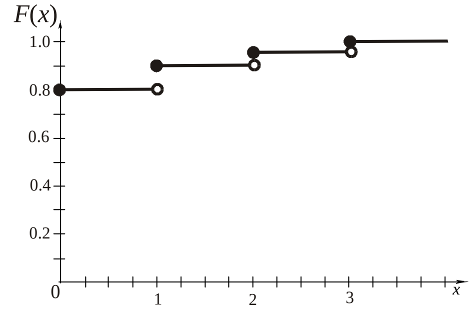
For Problem 2, $1-P(X<1)=1-P(X=0)$

$=1-0.7=0.3$

For Problem 5, $1-P(X<1)=1-0=1$

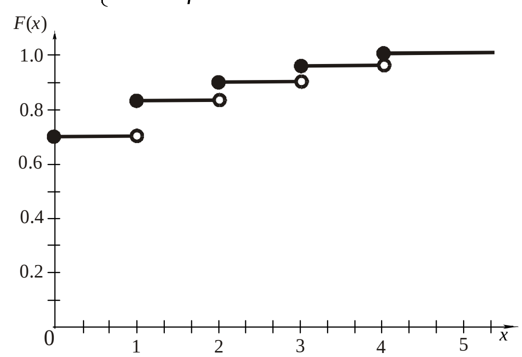
33.

$$F(x)=\begin{cases}0 & \text{if } x<0\\ 0.8 & \text{if } 0\le x<1\\ 0.9 & \text{if } 1\le x<2\\ 0.95 & \text{if } 2\le x<3\\ 1 & \text{if } x>3\end{cases}$$



34.

$$F(x)=\begin{cases}0 & \text{if } x<0\\ 0.7 & \text{if } 0\le x<1\\ 0.85 & \text{if } 1\le x<2\\ 0.9 & \text{if } 2\le x<3\\ 0.95 & \text{if } 3\le x<4\\ 1 & \text{if } x\ge 4\end{cases}$$



35. **a.** $P(Y<2)=P(Y\le 2)=F(2)=1$

b. $P(0.5<Y<0.6)=F(0.6)-F(0.5)$

$=\dfrac{1.2}{1.6}-\dfrac{1}{1.5}=\dfrac{1}{12}$

c. $f(y)=F'(y)=\dfrac{2}{(y+1)^2},\ 0\le y\le 1$

d. $E(Y)=\int_0^1 y\cdot\dfrac{2}{(y+1)^2}\,dy\approx 0.38629$

36. **a.** $P(Z > 1) = 1 - P(Z \le 1) = 1 - F(1)$
$= 1 - \frac{1}{9} = \frac{8}{9}$

b. $P(1 < Z < 2) = P(1 \le Z \le 2) = F(2) - F(1)$
$= \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$

c. $f(z) = F'(z) = \frac{2z}{9}, 0 \le z \le 3$

d. $E(Z) = \int_0^3 z \cdot \frac{2z}{9} dz = \left[\frac{2z^3}{27}\right]_0^3 = 2$

37. $E(X) = \int_0^4 x \cdot \frac{15}{512} x^2 (4-x)^2 \, dx = 2$
and $E(X^2) = \int_0^4 x^2 \cdot \frac{15}{512} x^2 (4-x)^2 \, dx$
$= \frac{32}{7} \approx 4.57$ using a CAS

38. $E(X^2) = \int_0^8 x^2 \cdot \frac{3}{256} x(8-x)\, dx = 19.2$ and
$E(X^3) = \int_0^8 x^3 \cdot \frac{3}{256} x(8-x)\, dx = 102.4$
using a CAS

39. $V(X) = E\left[(X-\mu)^2\right]$, where $\mu = E(X) = 2$
$V(X) = \int_0^4 (x-2)^2 \cdot \frac{15}{512} x^2 (4-x)^2 \, dx = \frac{4}{7}$

40. $\mu = E(X) = \int_0^8 x \cdot \frac{3}{256} x(8-x)\, dx = 4$
$V(X) = \int_0^8 (x-4)^2 \cdot \frac{3}{256} x(8-x)\, dx = \frac{16}{5}$

41. $E\left[(X-\mu)^2\right] = E(X^2 - 2X\mu + \mu^2)$
$= E(X^2) - E(2X\mu) + E(\mu^2)$
$= E(X^2) - 2\mu \cdot E(X) + \mu^2$
$= E(X^2) - 2\mu^2 + \mu^2$ since $E(X) = \mu$
$= E(X^2) - \mu^2$
For Problem 37, $V(X) = E(X^2) - \mu^2$ and using previous results, $V(X) = \frac{32}{7} - 2^2 = \frac{4}{7}$

5.8 Chapter Review

Concepts Test

1. False: $\int_0^{\pi} \cos x \, dx = 0$ because half of the area lies above the x-axis and half below the x-axis.

2. True: The integral represents the area of the region in the first quadrant if the center of the circle is at the origin.

3. False: The statement would be true if either $f(x) \ge g(x)$ or $g(x) \ge f(x)$ for $a \le x \le b$. Consider Problem 1 with $f(x) = \cos x$ and $g(x) = 0$.

4. True: The area of a cross section of a cylinder will be the same in any plane parallel to the base.

5. True: Since the cross sections in all planes parallel to the bases have the same area, the integrals used to compute the volumes will be equal.

6. False: The volume of a right circular cone of radius r and height h is $\frac{1}{3}\pi r^2 h$. If the radius is doubled and the height halved the volume is $\frac{2}{3}\pi r^2 h$.

7. False: Using the method of shells, $V = 2\pi \int_0^1 x(-x^2 + x) dx$. To use the method of washers we need to solve $y = -x^2 + x$ for x in terms of y.

8. True: The bounded region is symmetric about the line $x = \frac{1}{2}$. Thus the solids obtained by revolving about the lines $x = 0$ and $x = 1$ have the same volume.

9. False: Consider the curve given by $x = \frac{\cos t}{t}$, $y = \frac{\sin t}{t}, 2 \le t < \infty$.

10. False: The work required to stretch a spring 2 inches beyond its natural length is $\int_0^2 kx\, dx = 2k$, while the work required to stretch it 1 inch beyond its natural length is $\int_0^1 kx\, dx = \frac{1}{2}k$.

11. False: If the cone-shaped tank is placed with the point downward, then the amount of water that needs to be pumped from near the bottom of the tank is much less than the amount that needs to be pumped from near the bottom of the cylindrical tank.

12. False: The force depends on the depth, but the force is the same at all points on a surface as long as they are at the same depth.

13. True: This is the definition of the center of mass.

14. True: The region is symmetric about the point (π , 0).

15. True: By symmetry, the centroid is on the line $x=\frac{\pi}{2},$ so the centroid travels a distance of $2\pi\left(\frac{\pi}{2}\right)=\pi^2.$

16. True: At slice y, $\Delta A\approx(9-y^2)\Delta y$.

17. True: Since the density is proportional to the square of the distance from the midpoint, equal masses are on either side of the midpoint.

18. True: See Problem 30 in Section 5.6.

19. True: A discrete random variable takes on a finite number of possible values, or an infinite set of possible outcomes provided that these outcomes can be put in a list such as $\{x_1, x_2, \ldots\}$.

20. True: The computation of *E(X)* would be the same as the computation for the center of mass of the wire.

21. True: $E(X)=5\cdot 1=5$

22. True: If $F(x)=\int_A^x f(t)\,dt$, then $F'(x)=f(x)$ by the First Fundamental Theorem of Calculus.

23. True: $P(X=1)=P(1\le X\le 1)=\int_1^1 f(x)\,dx=0$

Sample Test Problems

1. $A=\int_0^1(x-x^2)dx=\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1=\frac{1}{6}$

2.
$$V=\pi\int_0^1(x-x^2)^2\,dx=\pi\int_0^1(x^2-2x^3+x^4)dx=\pi\left[\frac{1}{3}x^3-\frac{1}{2}x^4+\frac{1}{5}x^5\right]_0^1=\frac{\pi}{30}$$

3.
$$V=2\pi\int_0^1 x(x-x^2)dx=2\pi\int_0^1(x^2-x^3)dx=2\pi\left[\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1=\frac{\pi}{6}$$

4.
$$V=\pi\int_0^1\left[(x-x^2+2)^2-(2)^2\right]dx=\pi\int_0^1(x^4-2x^3-3x^2+4x)\,dx=\pi\left[\frac{1}{5}x^5-\frac{1}{2}x^4-x^3+2x^2\right]_0^1=\frac{7\pi}{10}$$

5.
$$V=2\pi\int_0^1(3-x)(x-x^2)dx=2\pi\int_0^1(x^3-4x^2+3x)\,dx=2\pi\left[\frac{1}{4}x^4-\frac{4}{3}x^3+\frac{3}{2}x^2\right]_0^1=\frac{5\pi}{6}$$

6.
$$\overline{x}=\frac{\int_0^1 x(x-x^2)dx}{\int_0^1(x-x^2)dx}=\frac{\left[\frac{1}{3}x^3-\frac{1}{4}x^4\right]_0^1}{\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1}=\frac{1}{2}$$

$$\overline{y}=\frac{\frac{1}{2}\int_0^1(x-x^2)^2\,dx}{\int_0^1(x-x^2)dx}=\frac{\frac{1}{2}\left[\frac{1}{3}x^3-\frac{1}{2}x^4+\frac{1}{5}x^5\right]_0^1}{\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1}=\frac{1}{10}$$

7. From Problem 1, $A=\frac{1}{6}$.

From Problem 6, $\overline{x}=\frac{1}{2}$ and $\overline{y}=\frac{1}{10}$.

$$V(S_1)=2\pi\left(\frac{1}{10}\right)\left(\frac{1}{6}\right)=\frac{\pi}{30}$$

$$V(S_2)=2\pi\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)=\frac{\pi}{6}$$

$$V(S_3)=2\pi\left(\frac{1}{10}+2\right)\left(\frac{1}{6}\right)=\frac{7\pi}{10}$$

$$V(S_4)=2\pi\left(3-\frac{1}{2}\right)\left(\frac{1}{6}\right)=\frac{5\pi}{6}$$

8. $8 = F(8) = 8k$, $k = 1$

a. $W = \int_2^8 x\,dx = \left[\frac{1}{2}x^2\right]_2^8 = \frac{1}{2}(64-4)$

$= 30$ in.-lb

b. $W = \int_0^{-4} x\,dx = \left[\frac{1}{2}x^2\right]_0^4 = 8$ in.-lb

9. $W = \int_0^6 (62.4)(5^2)\pi(10-y)dy$

$= 1560\pi\int_0^6 (10-y)dy = 1560\pi\left[10y - \frac{1}{2}y^2\right]_0^6$

$= 65{,}520\pi \approx 205{,}837$ ft-lb

10. The total work is equal to the work W_1 to pull up the object to the top without the cable and the work W_2 to pull up the cable.

$W_1 = 200 \cdot 100 = 20{,}000$ ft-lb

The cable weighs $\frac{120}{100} = \frac{6}{5}$ lb/ft.

$\Delta W_2 = \frac{6}{5}\Delta y \cdot y = \frac{6}{5}y\Delta y$

$W_2 = \int_0^{100} \frac{6}{5}y\,dy = \frac{6}{5}\left[\frac{1}{2}y^2\right]_0^{100}$

$= 6000$ ft-lb

$W = W_1 + W_2 = 26{,}000$ ft-lb

11. **a.** To find the intersection points, solve

$4x = x^2$.

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0, 4$

$A = \int_0^4 (4x - x^2)dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4$

$= \left(32 - \frac{64}{3}\right) = \frac{32}{3}$

b. To find the intersection points, solve

$\frac{y}{4} = \sqrt{y}$.

$\frac{y^2}{16} = y$

$y^2 - 16y = 0$

$y(y-16) = 0$

$y = 0, 16$

$A = \int_0^{16}\left(\sqrt{y} - \frac{y}{4}\right)dy = \left[\frac{2}{3}y^{3/2} - \frac{1}{8}y^2\right]_0^{16}$

$= \left(\frac{128}{3} - 32\right) = \frac{32}{3}$

12. $\overline{x} = \dfrac{\int_0^4 x(4x-x^2)dx}{\int_0^4 (4x-x^2)\,dx} = \dfrac{\int_0^4 (4x^2-x^3)\,dx}{\frac{32}{3}}$

$= \dfrac{\left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_0^4}{\frac{32}{3}} = \dfrac{\frac{64}{3}}{\frac{32}{3}} = 2$

$\overline{y} = \dfrac{\frac{1}{2}\int_0^4\left[(4x)^2 - (x^2)^2\right]dx}{\int_0^4 (4x-x^2)dx}$

$= \dfrac{\frac{1}{2}\int_0^4 (16x^2 - x^4)\,dx}{\frac{32}{3}}$

$= \dfrac{\frac{1}{2}\left[\frac{16}{3}x^3 - \frac{1}{5}x^5\right]_0^4}{\frac{32}{3}} = \dfrac{\frac{1024}{15}}{\frac{32}{3}} = \dfrac{32}{5}$

13. $V = \pi\int_0^4\left[(4x)^2 - (x^2)^2\right]dx$

$= \pi\int_0^4 (16x^2 - x^4)\,dx$

$= \pi\left[\frac{16}{3}x^3 - \frac{1}{5}x^5\right]_0^4 = \frac{2048\pi}{15}$

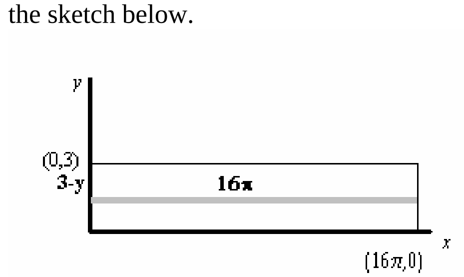
Using Pappus's Theorem:

From Problem 11, $A = \frac{32}{3}$.

From Problem 12, $\overline{y} = \frac{32}{5}$.

$V = 2\pi\overline{y}\cdot A = 2\pi\left(\frac{32}{5}\right)\left(\frac{32}{3}\right) = \frac{2048\pi}{15}$

14. **a.** (See example 4, section 5.5). Think of cutting the barrel vertically and opening the lateral surface into a rectangle as shown in the sketch below.



At depth $3 - y$, a narrow rectangle has width 16π, so the total force on the lateral surface is (δ = density of water = $62.4\ \text{lbs}/\text{ft}^3$)

$\int_0^3 \delta(3-y)(16\pi)\,dy = 16\pi\delta\int_0^3 (3-y)\,dy$

$= 16\pi\delta\left[3y - \frac{y^2}{2}\right]_0^3 = 16\pi\delta(4.5) \approx 14{,}114.55$ lbs.

b. All points on the bottom of the barrel are at the same depth; thus the total force on the bottom is simply the weight of the column of water in the barrel, namely

$F = \pi(8^2)(3)\delta \approx 37,638.8$ lbs.

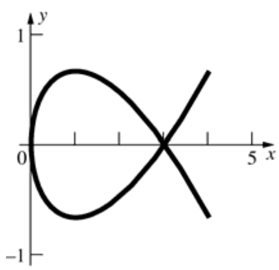
15. $\dfrac{dy}{dx} = x^2 - \dfrac{1}{4x^2}$

$$L = \int_1^3 \sqrt{1+\left(x^2 - \frac{1}{4x^2}\right)^2}\,dx$$

$$= \int_1^3 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}}\,dx = \int_1^3 \left(x^2 + \frac{1}{4x^2}\right)dx$$

$$= \left[\frac{1}{3}x^3 - \frac{1}{4x}\right]_1^3 = \left(9 - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{53}{6}$$

16.



The loop is $-\sqrt{3} \le t \le \sqrt{3}$. By symmetry, we can double the length of the loop from $t = 0$ to

$t = \sqrt{3}$, $\dfrac{dx}{dt} = 2t; \dfrac{dy}{dt} = t^2 - 1$

$$L = 2\int_0^{\sqrt{3}} \sqrt{t^4 + 2t^2 + 1}\,dt = 2\int_0^{\sqrt{3}} (t^2 + 1)dt$$

$$= 2\left[\frac{1}{3}t^3 + t\right]_0^{\sqrt{3}} = 4\sqrt{3}$$

17. $V = \int_{-3}^{3} \left(\sqrt{9 - x^2}\right)^2 dx = \int_{-3}^{3} (9 - x^2)\,dx$

$$= \left[9x - \frac{1}{3}x^3\right]_{-3}^{3} = (27 - 9) - (-27 + 9) = 36$$

18. $A = \int_a^b [f(x) - g(x)]dx$

19. $V = \pi\int_a^b \left[f^2(x) - g^2(x)\right]dx$

20. $V = 2\pi\int_a^b (x - a)[f(x) - g(x)]dx$

21. $M_y = \delta\int_a^b x[f(x) - g(x)]dx$

$M_x = \dfrac{\delta}{2}\int_a^b \left[f^2(x) - g^2(x)\right]dx$

22. $L_1 = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$

$L_2 = \int_a^b \sqrt{1 + [g'(x)]^2}\,dx$

$L_3 = f(a) - g(a)$

$L_4 = f(b) - g(b)$

Total length $= L_1 + L_2 + L_3 + L_4$

23. $A_1 = 2\pi\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx$

$A_2 = 2\pi\int_a^b g(x)\sqrt{1 + [g'(x)]^2}\,dx$

$A_3 = \pi\left[f^2(a) - g^2(a)\right]$

$A_4 = \pi\left[f^2(b) - g^2(b)\right]$

Total surface area $= A_1 + A_2 + A_3 + A_4$.

24. a. $P(X \ge 1) = P(1 \le X \le 2)$

$$= \int_1^2 \frac{1}{12}\left(8 - x^3\right)dx = \frac{1}{12}\left[8x - \frac{x^4}{4}\right]_1^2$$

$$= \frac{17}{48} \approx 0.354$$

b. $P(0 \le X < 0.5) = P(0 \le X \le 0.5)$

$$= \int_0^{0.5} \frac{1}{12}\left(8 - x^3\right)dx = \frac{1}{12}\left[8x - \frac{x^4}{4}\right]_1^2$$

$$= \frac{85}{256} \approx 0.332$$

c. $E(X) = \int_0^2 x \cdot \dfrac{1}{12}\left(8 - x^3\right)dx = \dfrac{1}{12}\left[4x^2 - \dfrac{x^5}{5}\right]_0^2$

$= 0.8$

d. $F(x) = \int_0^x \dfrac{1}{12}\left(8 - t^3\right)dt = \dfrac{1}{12}\left[8t - \dfrac{t^4}{4}\right]_0^x$

$$= \frac{1}{12}\left(8x - \frac{x^4}{4}\right) = \frac{2}{3}x - \frac{x^4}{48}$$

25. a. $P(X \le 3) = F(3) = 1 - \dfrac{(6-3)^2}{36} = \dfrac{3}{4}$

b. $f(x) = F'(x) = 2 \cdot \dfrac{1}{36}(6 - x) = \dfrac{6 - x}{18}$,

$0 \le x \le 6$

c. $E(X) = \int_0^6 x \cdot \left(\dfrac{6 - x}{18}\right)dx$

$$= \frac{1}{18}\left[3x^2 - \frac{x^3}{3}\right]_0^6 = 2$$

Review and Preview Problems

1. By the Power Rule

$$\int \frac{1}{x^2}dx = \int x^{-2}dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x} + C$$

2. By the Power Rule

$$\int \frac{1}{x^{1.5}}dx = \int x^{-1.5}dx = \frac{x^{-1.5+1}}{-1.5+1} = \frac{x^{-0.5}}{-0.5} = -\frac{2}{\sqrt{x}} + C$$

3. By the Power Rule

$$\int \frac{1}{x^{1.01}}dx = \int x^{-1.01}dx = \frac{x^{-1.01+1}}{-1.01+1} = \frac{x^{-0.01}}{-0.01} = -\frac{100}{x^{0.01}} + C$$

4. By the Power Rule

$$\int \frac{1}{x^{0.99}}dx = \int x^{-0.99}dx = \frac{x^{-0.99+1}}{-0.99+1} = \frac{x^{0.01}}{0.01} = 100x^{0.01} + C$$

5. $F(1) = \int_1^1 \frac{1}{t}dt = 0$

6. By the First Fundamental Theorem of Calculus

$$F'(x) = \frac{d}{dx}\int_1^x \frac{1}{t}dt = \frac{1}{x}$$

7. Let $g(x) = x^2$; then by the Chain Rule and problem 6,

$$D_x F(x^2) = D_x F(g(x)) = F'(g(x))g'(x)$$
$$= \left(\frac{1}{x^2}\right)(2x) = \frac{2}{x}$$

8. Let $h(x) = x^3$; then by the Chain Rule and problem 6,

$$D_x \int_1^{x^3} \frac{1}{t}dt = D_x F(h(x)) = F'(h(x))h'(x)$$
$$= \left(\frac{1}{x^3}\right)(3x^2) = \frac{3}{x}$$

9. **a.** $(1+1)^{1/1} = 2^1 = 2$

b. $(1+\frac{1}{5})^{1/\frac{1}{5}} = \left(\frac{6}{5}\right)^5 = 2.48832$

c. $(1+\frac{1}{10})^{1/\frac{1}{10}} = \left(\frac{11}{10}\right)^{10} \approx 2.593742$

d. $(1+\frac{1}{50})^{1/\frac{1}{50}} = \left(\frac{51}{50}\right)^{50} \approx 2.691588$

e. $(1+\frac{1}{100})^{1/\frac{1}{100}} = \left(\frac{101}{100}\right)^{100} \approx 2.704814$

10. **a.** $(1+\frac{1}{1})^1 = 2^1 = 2$

b. $(1+\frac{1}{10})^{10} = \left(\frac{11}{10}\right)^{10} \approx 2.593742$

c. $(1+\frac{1}{100})^{100} = \left(\frac{101}{100}\right)^{100} \approx 2.704814$

d. $(1+\frac{1}{1000})^{1000} = \left(\frac{1001}{1000}\right)^{1000} \approx 2.7169239$

11. **a.** $(1+\frac{1}{2})^{2/1} = \left(\frac{3}{2}\right)^2 = 2.25$

b. $(1+\frac{\frac{1}{5}}{2})^{2/\frac{1}{5}} = \left(1+\frac{1}{10}\right)^{10} = \left(\frac{11}{10}\right)^{10} \approx 2.593742$

c. $(1+\frac{\frac{1}{10}}{2})^{2/\frac{1}{10}} = \left(1+\frac{1}{20}\right)^{20} = \left(\frac{21}{20}\right)^{20} \approx 2.6533$

d. $(1+\frac{\frac{1}{50}}{2})^{2/\frac{1}{50}} = \left(1+\frac{1}{100}\right)^{100} = \left(\frac{101}{100}\right)^{100}$
≈ 2.70481

e. $(1+\frac{\frac{1}{100}}{2})^{2/\frac{1}{100}} = \left(1+\frac{1}{200}\right)^{200} = \left(\frac{201}{200}\right)^{200}$
≈ 2.71152

12. **a.** $(1+\frac{2}{1})^{1/2} = \sqrt{3} \approx 1.732051$

b. $(1+\frac{2}{10})^{10/2} = (1.2)^5 \approx 2.48832$

c. $(1+\frac{2}{100})^{100/2} = (1.02)^{50} \approx 2.691588$

d. $(1+\frac{2}{1000})^{1000/2} = (1.002)^{500} \approx 2.715569$

13. We know from trigonometry that, for any x and any integer k, $\sin(x+2k\pi) = \sin(x)$. Since

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \text{and} \quad \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2},$$

$$\sin(x) = \frac{1}{2} \text{ if } x = \frac{\pi}{6} + 2k\pi = \frac{12k+1}{6}\pi$$
$$\text{or } x = \frac{5\pi}{6} + 2k\pi = \frac{12k+5}{6}\pi$$

where k is any integer.

14. We know from trigonometry that, for any x and any integer k, $\cos(x+2k\pi)=\cos(x)$. Since $\cos(\pi)=-1$, $\cos(x)=-1$ if $x=\pi+2k\pi=(2k+1)\pi$ where k is any integer.

15. We know from trigonometry that, for any x and any integer k, $\tan(x+k\pi)=\tan(x)$. Since $\tan\left(\frac{\pi}{4}\right)=1$, $\tan(x)=1$ if $x=\frac{\pi}{4}+k\pi=\frac{4k+1}{4}\pi$ where k is any integer.

16. Since $\sec(x)=\frac{1}{\cos(x)}$, $\sec(x)$ is never 0.

17. In the triangle, relative to θ,
$opp=\sqrt{x^2-1}$, $adj=1$, $hypot=x$ so that

$$\sin\theta=\frac{\sqrt{x^2-1}}{x}\quad \cos\theta=\frac{1}{x}\quad \tan\theta=\sqrt{x^2-1}$$

$$\cot\theta=\frac{1}{\sqrt{x^2-1}}\quad \sec\theta=x\quad \csc\theta=\frac{x}{\sqrt{x^2-1}}$$

18. In the triangle, relative to θ,
$opp=x$, $adj=\sqrt{1-x^2}$, $hypot=1$ so that

$$\sin\theta=x\quad \cos\theta=\sqrt{1-x^2}\quad \tan\theta=\frac{x}{\sqrt{1-x^2}}$$

$$\cot\theta=\frac{\sqrt{1-x^2}}{x}\quad \sec\theta=\frac{1}{\sqrt{1-x^2}}\quad \csc\theta=\frac{1}{x}$$

19. In the triangle, relative to θ,
$opp=1$, $adj=x$, $hypot=\sqrt{1+x^2}$ so that

$$\sin\theta=\frac{1}{\sqrt{1+x^2}}\quad \cos\theta=\frac{x}{\sqrt{1+x^2}}\quad \tan\theta=\frac{1}{x}$$

$$\cot\theta=x\quad \sec\theta=\frac{\sqrt{1+x^2}}{x}\quad \csc\theta=\sqrt{1+x^2}$$

20. In the triangle, relative to θ,
$opp=\sqrt{1-x^2}$, $adj=x$, $hypot=1$ so that

$$\sin\theta=\sqrt{1-x^2}\quad \cos\theta=x\quad \tan\theta=\frac{\sqrt{1-x^2}}{x}$$

$$\cot\theta=\frac{x}{\sqrt{1-x^2}}\quad \sec\theta=\frac{1}{x}\quad \csc\theta=\frac{1}{\sqrt{1-x^2}}$$

21. $y'=xy^2 \;\rightarrow\; dy=xy^2dx$

$$\frac{1}{y^2}dy=xdx$$

$$\int\frac{dy}{y^2}=\int xdx$$

$$-\frac{1}{y}=\frac{1}{2}x^2+C$$

When $x=0$ and $y=1$ we get $C=-1$. Thus,

$$-\frac{1}{y}=\frac{1}{2}x^2-1=\frac{x^2-2}{2}$$

$$y=-\frac{2}{x^2-2}$$

22. $y'=\frac{\cos x}{y} \;\rightarrow\; dy=\frac{\cos x}{y}dx$

$$y\,dy=\cos x\,dx$$

$$\int ydy=\int\cos x\,dx$$

$$\frac{1}{2}y^2=\sin x+C$$

When $x=0$ and $y=4$ we get $C=8$. Thus,

$$\frac{1}{2}y^2=\sin x+8$$

$$y^2=2\sin x+16$$